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Dynamics of the periodic type-K competitive Kolmogorov systems ☆

Mats Gyllenberg[a], Yi Wang[b,*]

[a] *Department of Mathematics and Statistics, University of Helsinki, FIN-00014 Helsinki, Finland*
[b] *Department of Mathematics, University of Turku, FIN-20014 Turku, Finland*

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Abstract

For time-periodic dissipative and irreducible type-K competitive Kolmogorov systems, it is proved that there is a canonically defined countable family $\mathscr{F}$ of unordered, disjoint invariant sets with the property that, for every persistent trajectory whose ω-limit set is not a cycle, there exists a unique trajectory in some element of $\mathscr{F}$ such that these two trajectories are asymptotic and the corresponding points in these two trajectories are K-related.

Keywords: Time-periodic type-K competitive system; Poincaré map; Persistence; Invariant order decomposition; Exponential separation; Pesin's theory

1. Introduction

Consider a community of n interacting species modeled by the Kolmogorov equations

$$\begin{aligned} &\dot{x}_i = x_i f_i(t, x), \quad x_i \geqslant 0, \quad 1 \leqslant i \leqslant n, \\ &x = (x_1, x_2, \ldots, x_n), \end{aligned} \tag{1.1}$$

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* Corresponding author. Fax: +358-2-333-6595. Supported by the Academy of Finland and partially by NSF of Anhui Province, China. On leave from Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, PR China.
E-mail address: matsgyl@utu.fi, yiwang@utu.fi (Y. Wang).

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where $f(t,x) = (f_1,\dots,f_n)$ is defined and continuous, together with its first and second derivatives with respect to x, for $(t,x) \in \mathbb{R} \times U$, where U is an open subset of $\mathbb{R}^n$ containing $C = \{x : x_i \geqslant 0 \text{ for all } i\}$. Here x_i is the density and $f_i(t,x)$ is the per capita growth rate of species i. Note that f_i not only depends on the densities of the interacting populations, but also fluctuates with time. The interpretation requires us to restrict our attention to the closed positive cone C. System (1.1) is called *competitive* if all off-diagonal elements of the Jacobian matrix $D_x f(t,x)$ are non-positive at each $(t,x) \in \mathbb{R} \times C$ [14].

If f is independent of time, the well-known construction of Smale [27] showed that, contrary to the popular belief in early 1970s, mathematical models of competition between species could lead to differential equations with extremely complicated dynamics. This in turn led to a famous series of papers by Hirsch [7–10] who showed that the dynamics of competitive systems can be no worse than that of completely general systems of one less dimension. One of the most exciting fallouts of these results was to the study of n-dimensional dissipative competitive systems. More precisely, under the assumptions of competition, dissipation and irreducibility, Hirsch [9] proved that there is a canonically defined countable family of unordered, disjoint Lipschitz open $(n-1)$-cells such that every non-convergent persistent trajectory is asymptotic to a trajectory in one of the cells. In particular, if the origin is a repeller and the community matrices are strictly negative, then there exists a balanced attractor, called carrying simplex (see [11,43]), attracting all nontrivial orbits. Recently, by appealing to Takáč's invariant order decomposition methods (see [36,37]), Wang and Jiang [41,42] have generalized this well-known result of Hirsch's to Kolmogorov competitive mappings and the time-periodic competitive system (1.1). Of course, the system is called time-periodic of period τ if

$$f_i(t+\tau,x) = f_i(t,x) \quad \text{for all } (t,x) \in \mathbb{R} \times C,\ i = 1,\dots,n. \tag{1.2}$$

Smith [29] developed Hirsch's theory on competitive and monotone systems and introduced the so-called type-K monotone Kolmogorov systems. System (1.1) is called *type-K monotone* if there exists an integer k satisfying $1 \leqslant k < n$ such that, at each $(t,x) \in \mathbb{R} \times C$, the Jacobian matrix $D_x f(t,x)$ of f has the structure

$$\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \tag{1.3}$$

where A is a $k \times k$ matrix, D is a $(n-k) \times (n-k)$ matrix, $B, C \geqslant 0$, and each off-diagonal element of A and D is nonnegative. Of course, each matrix A, B, C, D depends on (t,x). As a matter of fact, the system (1.1) being type-K monotone implies that it is monotone with respect to an order on C. To see this, let K be the closed cone $\{x \in \mathbb{R}^n : x_i \geqslant 0,\ 1 \leqslant i \leqslant k;\ x_j \leqslant 0,\ k+1 \leqslant j \leqslant n\}$, Smith [29] introduced an order in C such that

$$x \leqslant_K y \quad \Leftrightarrow y - x \in K,$$

$$x \ll_K y \Leftrightarrow y - x \in \operatorname{Int} K,$$

$$x <_K y \Leftrightarrow y - x \in K \text{ and } y \neq x,$$

and proved that the system is type-K monotone in the sense that $\xi_1 \leqslant_K (\ll_K, <_K)\xi_2$ implies $\varphi(t,s,\xi_1) \leqslant_K (\ll_K, <_K)\varphi(t,s,\xi_2)$ for $t \geqslant s$, where $\varphi(t,s,\xi)$ is the unique solution of (1.1) satisfying the initial value $\varphi(s,s,\xi) = \xi$. For the autonomous case, Smith [29] established the uniform persistence of all species. Thereafter, the coexistence, extinction and global attractivity of the autonomous and time-periodic type-K monotone Kolmogorov systems have been the target of extensive investigations (see [16,17,33,39]).

System (1.1) is called *type-K competitive* if at each $(t,x) \in \mathbb{R} \times C$, $-D_x f(t,x)$ has the form (1.3). Many mathematical models such as the growth of phytoplankton in a chemostat [34], some predator–prey models [5], models of the signaling systems of the slime mold [4,32], the Field–Noyes models of Zhabotinski reaction (see, Murray [22]), and all kinds of competitor–competitor-mutualist models (see [24,35,44,45,2]) are type-K competitive according to our definition.

Recently, Liang and Jiang [18] have investigated the large-time behavior of the autonomous type-K competitive Kolmogorov system (1.1). Under the additional assumptions of dissipation and irreducibility, they proved that there is a canonically defined countable family of disjoint invariant $(n-1)$ cells which attract all non-convergent persistent trajectories. Note that this result is absolutely not a direct corollary of Hirsch's result in [9] because of the special structure of the type-K competitive Kolmogorov system. The key approaches in [18] are the α-limit dichotomy of the type-K competitive Kolmogorov system and the construction of two very special $(n-1)$-dimensional invariant sets, $V_-(\infty)$ and $V_+(\infty)$.

Our focus in this paper is on time-periodic type-K competitive Kolmogorov systems (1.1)–(1.2). The discrete-time dynamics of the Poincaré map T associated with (1.1)–(1.2) will be studied. We shall generalize the results of [18] to the n-dimensional time-periodic type-K competitive systems (1.1)–(1.2). More precisely, under the assumptions of dissipation and irreducibility, we shall prove that there is a canonically defined countable family $\mathcal{F}$ of unordered, disjoint invariant sets with the property that, for every persistent trajectory, whose ω-limit set is not a cycle, there exists a unique trajectory in some element of $\mathcal{F}$ such that these two trajectories are asymptotic and the corresponding points in these two trajectories are K-related. Note that in the time-periodic case the α-limit dichotomy does not hold any more. Moreover, to the best of our knowledge, the uniqueness of the asymptotic phase in this case is for the first time considered in the present paper. The approach in this work, which is totally different from that of Liang and Jiang's, is due to the second author's earlier joint work [41,42] with Jiang, but it is more difficult to construct the family $\mathcal{F}$ and one needs more techniques and tools to prove that the ω-limit sets are in some element of such a family. The proof of the uniqueness of the asymptotic phase is mainly based on exponential separation (cf. [20, Appendix]) and Pesin's Theory.

The paper is organized as follows. In Section 2, we agree on some notations, give important definitions and state some fundamental results which will be important to our proofs. The main result and its proof are given in Section 3.

2. Notations and fundamental results

Let $C = \{x \in \mathbb{R}^n : x \geqslant 0\}$ be the usual nonnegative orthant. The interior of C is $C^o = \{x \in \mathbb{R}^n : x \gg 0\}$ and the boundary of C is ∂C. Given an integer $1 \leqslant k < n$, define a cone K in $\mathbb{R}^n$ as $K = \{x \in \mathbb{R}^n : x_i \geqslant 0,\ 1 \leqslant i \leqslant k;\ x_j \leqslant 0,\ k+1 \leqslant j \leq n\}$, we write $x \leqslant_K y$ if $y - x \in K$, $x <_K y$ if $y - x \in K$ and $y \neq x$, and $x \ll_K y$ if $y - x \in \operatorname{Int} K$, the interior of K. Given $\emptyset \neq I \subset N := \{1, 2, \ldots, n\}$, let $H_I = \{x \in \mathbb{R}^n : x_j = 0 \text{ for } j \notin I\}$. We also let $H_I^+ = C \cap H_I$ and $\dot{H}_I^+ = \{x \in H_I^+ : x_i > 0 \text{ for } i \in I\}$. For sets $A, B \subset C$ we write $A \preceq_K (\prec_K, \prec\prec_K) B$ if given any $a \in A$, there exists some $b \in B$ such that $a \leqslant_K (<_K, \ll_K) b$ and given any $y \in B$ there is some $x \in A$ such that $x \leqslant_K (<_K, \ll_K) y$. Related notations such as $x \geqslant_K y$, $A \succeq_K B$, etc., have the natural meanings. For any points $x \ll_K y$ in $\mathbb{R}^n$ we define the *open order interval* $[[x, y]]_K = \{z \in \mathbb{R}^n : x \ll_K z \ll_K y\}$ and the *closed order interval* $[x, y]_K = \{z \in \mathbb{R}^n : x \leqslant_K z \leqslant_K y\}$. We also let $[[x, y]_K = \{z \in \mathbb{R}^n : x \ll_K z \leqslant_K y\}$ and $[x, +\infty)_K = \{y \in C : y \geqslant_K x\}$, and related notations such as $(-\infty, x]_K$, $[x, y]]_K$ and $[[x, \infty)$, etc., have the natural meanings. A set in $\mathbb{R}^n$ is *order convex* if it contains the order closed intervals defined by each pair of its elements. A subset Y of X is called *lower closed* if $(-\infty, b]_K \subset Y$ whenever $b \in Y$; and *upper closed* if $[a, +\infty)_K \subset Y$ whenever $a \in Y$. A point z is in the *lower boundary* $\partial_- S$ of a set $S \subset \mathbb{R}^n$ provided there is a sequence $\{y_n\}$ in S converging to z with $y_n \gg_K z$(denoted by $y_n \downarrow z$), but no sequence $\{x_n\}$ in S converging to z with $x_n \ll_K z$(denoted by $x_n \uparrow z$). The upper boundary $\partial_+ S$ is defined analogously. An *open k-cell* in a space X is a subset homeomorphic to $\mathbb{R}^k$. A *closed k-cell* is a subset homeomorphic to the closed unit ball in $\mathbb{R}^k$. Suppose that X is a metrizable topological vector space together with a closed convex cone V_+. Then (X, V_+) is called *strongly ordered* if $\operatorname{Int} V_+ \neq \emptyset$. A pair of (A, B) of subsets A, B of a strongly ordered topological vector space X is called an *invariant order decomposition* of X if it has the following properties: (i) $A \neq \emptyset$ and $B \neq \emptyset$; (ii) A and B are closed; (iii) A is lower closed and B is upper closed; (iv) $A \cup B = X$; (v) $\operatorname{Int}(A \cap B) = \emptyset$ and (vi) $TA \subset A, T(B) \subset B$. It is easy to see that $H = A \cap B$ never contains two distinct points x, y such that $y - x \in \operatorname{Int} V_+$. If A is a subset of topological space X, $\overline{A}$ denotes the closure of A in X. The boundary of A relative to X is denoted by $\partial_X A$, or ∂A if $X = C$.

We write $\varphi(t, s, x_0)$ for the solution map of (1.1), that is, $\varphi(\cdot, s, x_0)$ is the unique solution of (1.1) satisfying $\varphi(s, s, x_0) = x_0$. We will assume without further mentioning that the domain of $\varphi(t, s, x_0)$ includes $[s, +\infty)$ in case $x_0 \in C$. In particular, when $s = 0$, we suppose that $(l_{x_0}, +\infty)$ is the maximal existence interval of the solution $\varphi(t, 0, x_0)$ in case $x_0 \in C$.

It is convenient to define the fundamental object of study in this paper, namely the Poincaré map for the τ-periodic system (1.1)–(1.2). Let

$$T(x) = \varphi(\tau, 0, x)$$

for $x \in C$. It is known that T is a C^1-diffeomorphism onto $T(C)$ which is orientation preserving. The set $A \subset C$ is called *invariant* (resp. *positively invariant*) if $TA = A$

(resp. $TA \subset A$). Let $x \in C$. Then either there exists some $N \in \mathbb{N}$ such that $T^{-n}x \in TC$ for $0 \leqslant n \leqslant N$ but $T^{-(N+1)}x \notin TC$, or $T^{-n}x \in TC$ for any $n \in \mathbb{N}$. In the first case, we say that such an x does not have a full backward orbit. The orbit of any $x \in C$ in C is defined by $O(x) = \{T^m x : T^m x \geqslant 0 \text{ and } m \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers. Obviously, $O(x)$ is positively invariant. The *ω-limit set* of x is defined by $\omega(x) = \{y \in C : T^{n_k}x \to y(k \to \infty)$ for some sequence $n_k \to +\infty$ in $\mathbb{Z}\}$ and the *α-limit set* of x by $\alpha(x) = \{y \in C : T^{-n_k}x \to y(k \to \infty)$ for some sequence $n_k \to +\infty$ in $\mathbb{Z}\}$. Note that if $\overline{O(x)}$ is compact in C, then ω-limit set of x is nonempty and invariant. Furthermore, the α-limit set of x is nonempty and invariant provided x has a full backward orbit and $\overline{O(x)}$ is compact in C. Given any $x \in C^o$, x is called have a *persistent orbit* if there exists some $\delta > 0$ such that $\liminf_{n\to+\infty} T_i^n(x) \geqslant \delta$ for all $i = 1, 2, \ldots, n$.

A point p is called an $(m-)$ *periodic point* of T if $T^m p = p$ and $T^l p \neq p$ for $0 < l < m$. We call $O(p)$ a cycle, or an *m-cycle*. If $Tp = p$, then we say p is a *fixed point*. We also denote by $\mathrm{Per}(T)$ the set of periodic points of T in C and $Fix(T)$ the set of fixed points of T in C. Given an m-cycle $O(p)$, we denote $\lambda_1(p) = \rho(DT^{-m}(p))$, where $\rho(DT^{-m}(p))$ is the spectral radius of the linear mapping $DT^{-m}(p)$. If $\lambda_1(p) \leqslant 1$, we call $O(p)$ is a *linearly stable* cycle of T^{-1}.

The Kolmogorov system (1.1) is called a *type-K competitive* if the Jacobian matrix $-D_x f(t,x)$ satisfies (1.3) for each $(t,x) \in \mathbb{R} \times C$. The Poincaré map T associated with time-periodic type-K competitive Kolmogorov system (1.1)–(1.2) satisfies

(A1) If $x, y \in C$ and $Tx <_K Ty$, then $x <_K y$ (we call such a map T *type-K competitive*).

(A1′) If $x_1, x_2 \in C$ and $Tx_1 = y_1, Tx_2 = y_2$ with $y_1 <_K y_2$, then $[y_1, y_2]_K \subseteq T[x_1, x_2]_K$.

Conditions (A1) and (A1′) are essentially due to Smith [28] with a change of the cone K. Since T is the Poincaré map associated with the ODE system (1.1)–(1.2), it is not difficult to show that (A1) implies (A1′) and we refer this to Smith [28]. By the well-known Kamke–Müller Theorem (see, [1,31]), we can also obtain that T has the following property **(S)**. *If $Tx \ll_K Ty$, then $x \ll_K y$.*

We call system (1.1) *irreducible in each face* if, for each $I \subset N, I \neq \emptyset$, the matrix $(D_x f(t,x))_{I\times I}$ is irreducible at each $(t,x) \in \mathbb{R} \times H_I^+$. Recall that a matrix A is irreducible if there is no permutation matrix R such that $R^{-1}AR = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ with square matrices A_1, A_2. Suppose that the periodic system (1.1)–(1.2) is type-K competitive and irreducible in each face, then the Poincaré map T satisfies

(A2) For each $I \subset N, I \neq \emptyset$ and $x, y \in H_I^+$, if $Tx <_K Ty$ and either x or y is in $\dot{H}_I^+$, then $x_i < y_i$ for all $i \in I \cap \{1, 2, \ldots, k\}$ and $x_j > y_j$ for all $j \in I \setminus \{1, 2, \ldots, k\}$.

From point of view of applications to the real ecosystems, it is also reasonable to assume that the Poincaré map T is *Dissipative*, i.e.,

(A3) There is a compact invariant set Γ, called the *fundamental attractor*, which uniformly attracts each compact set of the initial values.

Note that $f(t,x) = (f_1, \ldots, f_n)$ is continuous, together with its first and second derivatives with respect to x, for $(t,x) \in \mathbb{R} \times U$, where U is an open subset of $\mathbb{R}^n$ containing C. Then the Poincaré map T has the following smoothness:

(A4) T is an injective C^2-diffeomorphism onto $T(U)$.

Now we state several known results which will be important in the proof of the main result.

Proposition 2.1 (Wang and Jiang). *Let (X, K) be a strongly ordered metrizable topological space. Let $T : X \to X$ be a continuous type-K competitive map satisfying* (S). *Assume that G is a non-empty, unordered with respect to $\ll_K$, invariant subset of X. Then there exists an invariant order decomposition (Y, Z) of X such that $G \subset H = Y \cap Z$.*

Proposition 2.2. *Let $V \subset \mathbb{R}^n$ be an open $(n-1)$-cell which is unordered with respect to $\ll_K$. Then for any $u \gg_K 0$ the map $h : V \times \mathbb{R} \to \mathbb{R}^n$ defined by $(x, \lambda) \to x + \lambda u$ is a homeomorphism onto an open n-cell.*

Proposition 2.3 (Non-ordering of limit-sets). *Any ω- or α- limit set $L \subset C^o$ cannot contain two points related by $<_K$.*

Proposition 2.4. *Suppose that $x <_K y$ and x, y have full backward orbits. Then $\alpha(x) \preceq_K \alpha(y)$.*

Proposition 2.1 can be found in [40]. The definition of invariant order decomposition are due to Takáč [36,37]. Proposition 2.2 is essentially due to Hirsch [9, Proposition 2.7]. Propositions 2.3 and 2.4 can be found in [28] and [36,37], respectively.

The proof of the uniqueness of the asymptotic phase requires measure-theoretic methods. Hence, before closing this section, we review some definitions and theorems from ergodic theory. Given a Borel probability measure μ on an invariant set S, *the support* of μ, denoted $\mathrm{supp}(\mu)$, is the smallest closed set whose complement has measure 0. A Borel probability measure μ is called *invariant* for the map T, provided that $\mu(B) = \mu(T^{-1}B)$ for every Borel set $B \subset S$. An invariant measure μ is called *ergodic* provided that $\mu(B) = 0$ or 1 for any invariant Borel set $B \subset S$. Let $\mathcal{M}_{erg}(S)$ denote the subset of ergodic measures with support in S.

Fix a $\mu \in \mathcal{M}_{erg}(S)$, we recall the Oseledec's Multiplicative Ergodic Theorem [15,19]: there exists a finite set of real numbers $\mathcal{L} \subset \mathbb{R}$ and an invariant Borel set $O \subset \mathrm{supp}(\mu)$ with $\mu(O) = 1$ such that for each $x \in O$ there is a splitting $V = \bigoplus_{\lambda \in \mathcal{L}} E^{\lambda}(x)$ satisfying $DT(x)E^{\lambda}(x) = E^{\lambda}(Tx)$ and $\lim_{n\to\pm\infty} \frac{1}{n} \log \|DT^n(x)v\| = \lambda$ for all $v \in E^{\lambda}(x) \setminus \{0\}$. The set $\mathcal{L}$ is the set of *Lyapunov exponents* for μ and the set O is called the set of *Oseledec regular points* for μ.

Because of hypothesis (A4), we can use Pesin's Theory. In the sequel we will need the following fact: For μ as above, there is an invariant Borel set $Q \subset O$, $\mu(Q) = 1$ ($x \in Q$ is called a P-regular point), with the property that for any $x \in Q$ and any $k \in \{1, \ldots, l\}$ for which $\lambda_k < 0$ (resp. $\lambda_k > 0$) there exists a C^1 immersed disc D tangent at x to $\bigoplus_{j=1}^{k} E^{\lambda_j}(x)$(resp. $\bigoplus_{j=k}^{l} E^{\lambda_j}(x)$) such that $\|T^n x - T^n y\| \to 0$ (resp. $\left\|T^{-n} x - T^{-n} y\right\| \to 0$) as $n \to \infty$, for all $y \in D$ (see [25]).

3. The main result and its proof

Our main result is as follows

Theorem A. *Let $T: C \to C$ be the Poincaré map associated with system* (1.1)–(1.2). *Assume that hypotheses* (A1)–(A4) *hold. Then there is a family $\mathcal{F} = \{M_i\}_{i \in \mathbb{N}}$ of disjoint positively invariant sets in C having the following properties*:

(a) *Each $M_i (i \geqslant 2)$ is invariant, while M_0, M_1 is weakly invariant, i.e., M_l is positively invariant and if $x \in M_l \cap T(C)$, then $T^{-1}x \in M_l$ for $l = 0, 1$.*

(b) *Each M_i is the union of finitely many disjoint open $(n-1)$-cells and each open $(n-1)$-cell is a Lipschitz submanifold.*

(c) *For every M_i in $\mathcal{F}$, no two points of M_i are related by $\ll_K$ and no two points of $M_i \cap C^o$ are related by $<_K$.*

(d) *Given any $M_i \in \mathcal{F}$, there is an invariant order decomposition(Y, Z) of C such that $M_i \subset H = Y \cap Z$. Furthermore, H is an $(n-1)$-dimensional Lipschitz submanifold.*

(e) *Every persistent orbit which is not convergent to a cycle is asymptotic to a unique orbit in some M_i such that the corresponding points on these two orbits are K-related. More precisely, suppose that the orbit of x does not converge to a cycle and $\omega(x) \subset C^o$, then $\omega(x) \subset M_i$ for some i. If $x \notin M_i$, then there exists some $l \in \mathbb{N}$ such that for such l, there is a unique $y_l \in M_i \cap C^o$ with $\left\| T^{n+l}x - T^n y_l \right\| \to 0$ as $n \to \infty$ and either $T^{n+l}x \ll_K T^n y_l$ or $T^{n+l}x \gg_K T^n y_l$ for all $n \in \mathbb{N}$. In particular, if $i \geqslant 2$, then $l = 0$.*

Remark 3.1. Theorem A generalizes Theorems 3.4 and 3.9 in [18] for *n*-dimensional competitive autonomous systems to the *n*-dimensional competitive τ-periodic systems and describes explicitly where the positive ω-limit sets lie and gives more topological and structural information of the invariant sets constituting the family $\mathcal{F}$. Theorem A is also, to the best of our knowledge, the first one which shows the uniqueness of the asymptotic phase in this case. Moreover, even the proof of the existence of the family $\mathcal{F}$ (not including the proof of the uniqueness of the asymptotic phase) in Theorem A is still absolutely NOT a direct corollary of Theorem 3.1 in [41]. An examination of the proofs of Theorem 3.1 in [41] suggests that they strongly depend on the compatibility of the normal order relation $\leq$ with the phase space C, i.e., in [41], we can define the set $\omega(x)_+ = \{y \in \Gamma : y \geqslant z \text{ for some } z \in \omega(x)\}$($y \geqslant z$ means $y - z \in C$), which is order-convex, compact and belongs to C^o. Therefore the well-known Tereščák's result (cf. [38]) on $\omega(x)_+$ can be used as a key to the proof of Theorem 3.1 in [41]. In contrast, in our Theorem A, we cannot define $\omega(x)_+$ similarly as $\omega(x)_+ = \{y \in \Gamma : y \geqslant_K z \text{ for some } z \in \omega(x)\}$. The main reason is that $\omega(x)_+$ does not belong to C^o anymore, and therefore Tereščák's result is invalid in this case. As a consequence, we need new methods to solve this problem.

Hereafter, we always assume that hypotheses (A1)–(A4) hold for the Poincaré map T of the time-periodic system (1.1)–(1.2). The proof of Theorem A is conveniently broken into several propositions. We begin with some definitions which are very useful.

Suppose that p is an m-periodic point. Let $O(p) = \{p_0, p_1, \ldots, p_{m-1}\}$ be an m-cycle with $T^i p_j = p_{(i+j)_{\mathrm{mod}\, m}}$ and $p_0 = p$. For $i \in \{0, 1, \ldots, m-1\}$, we define

$$M(p_i) = \{x \in C : T^{-nm}x \to p_i \text{ as } n \to +\infty\},$$

$$M_-(p_i) = \{x \in M(p_i) : T^{-nm}x \ll_K p_i \text{ for sufficiently large } n \in \mathbb{N}\},$$

$$M_+(p_i) = \{x \in M(p_i) : T^{-nm}x \gg_K p_i \text{ for sufficiently large } n \in \mathbb{N}\},$$

$$V_-(p_i) = \partial_-(M_-(p_i)), \qquad V_+(p_i) = \partial_+(M_+(p_i))$$

and

$$M(O(p)) = \bigcup_{i=0}^{m-1} M(p_i),$$

$$M_-(O(p)) = \bigcup_{i=0}^{m-1} M_-(p_i), \quad M_+(O(p)) = \bigcup_{i=0}^{m-1} M_+(p_i),$$

$$V_-(O(p)) = \bigcup_{i=0}^{m-1} V_-(p_i), \quad V_+(O(p)) = \bigcup_{i=0}^{m-1} V_+(p_i).$$

We then have the following proposition.

Proposition 3.2. *Let* $p, q \in \mathrm{Per}(T) \cap C^o$ *and* $O(p) \neq O(q)$. *Then we have*

(a) $M(O(p))$, $M_-(O(p))$, $M_+(O(p))$, $V_-(O(p))$ *and* $V_+(O(p))$ *are invariant*;

(b) $M(O(p))$, $M_-(O(p))$ *and* $M_+(O(p))$ *are order convex*; $M_-(O(p))$ *and* $M_+(O(p))$ *is an open set in* C;

(c) $M_-(O(p)) \cap M_-(O(q)) = \emptyset$ *and* $V_-(O(p)) \cap V_-(O(q)) = \emptyset$;

(d) $V_-(O(p))$ *is unordered with respect to* $\ll_K$, *and* $V_-(O(p)) \cap C^o$ *is unordered with respect to* $<_K$;

(e) *there is an invariant order decomposition* (Y, Z) *of* C *such that* $V_-(O(p)) \subset H = Y \cap Z$, *where* H *is an* $(n-1)$*-dimensional Lipschitz submanifold*;

(f) *if* $M_-(O(p)) \neq \emptyset$, *then* $M_-(O(p))$ *is the union of finite disjoint open n-cells and* $V_-(O(p))$ *is the union of finite disjoint open* $(n-1)$*-cells*;

(g) *results analogous to* (c)–(f) *hold for* $M_+(O(p))$ *and* $V_+(O(p))$.

The proof of Proposition 3.2 is completely similar to the proof of Proposition 3.3 in [41]. We omit it here.

Now, we focus on the fundamental attractor Γ. Obviously, the fundamental attractor Γ of T is the fundamental repellor of T^{-1}. In terms of T^{-1}, Γ is characterized as the set of points with bounded orbits, while $x \in C \setminus \Gamma$ if and only if either x does not have a full backward orbit, or $\left|T^{-n}x\right| \to +\infty$ as $n \to +\infty$. Therefore, it is easy to see that Γ is order convex. Hereafter, we also assume that $\Gamma \cap C^o \neq \emptyset$, since we shall discuss the persistent orbits below.

Define

$$\begin{aligned}
R(\infty) &= C \setminus \Gamma,\\
\hat{R}_-(\infty) &= \{x \in C^o \cap R(\infty) : \sup_{t \in (l_x, 0]} \varphi_i(t, 0, x) < +\infty \text{ for } i = k+1, \ldots, n\},\\
\hat{R}_+(\infty) &= \{x \in C^o \cap R(\infty) : \sup_{t \in (l_x, 0]} \varphi_i(t, 0, x) < +\infty \text{ for } i = 1, \ldots, k\},\\
R_-(\infty) &= \text{ the interior of } \hat{R}_-(\infty) \text{ in } \mathbb{R}^n,\\
R_+(\infty) &= \text{ the interior of } \hat{R}_+(\infty) \text{ in } \mathbb{R}^n,\\
V_-(\infty) &= \partial_- R_-(\infty) \cap C^o,\\
V_+(\infty) &= \partial_+ R_+(\infty) \cap C^o.
\end{aligned}$$

The following proposition states the fundamental properties of the sets defined above.

Proposition 3.3. *We have*

(1) $\hat{R}_-(\infty)$ *is non-empty, order convex, upper closed in* C^o *and weakly invariant in the sense of* (a) $T\hat{R}_-(\infty) \subset \hat{R}_-(\infty)$ *and* (b) $T^{-1}x \in \hat{R}_-(\infty)$ *if* $x \in \hat{R}_-(\infty) \cap T(C)$;

(2) $R_-(\infty)$ *is a non-empty, order convex, upper closed in* C^o *and weakly invariant open n-cell*;

(3) $V_-(\infty)$, *which is closed in C, is a weakly invariant open* $(n-1)$*-cell and no two points of* $V_-(\infty)$ *are related by* $<_K$;

(4) *results analogous to* (1)–(3) *hold for* $\hat{R}_+(\infty)$, $R_+(\infty)$ *and* $V_+(\infty)$.

Proof. We prove only (1)–(3). The proof of (4) is similar. (1): Fix an $x \in C^o \cap \Gamma$. Since Γ is compact, $[[x, +\infty)_K \cap R(\infty) \neq \emptyset$. Let $y \in [[x, +\infty)_K \cap R(\infty) \cap C^o$, then $\varphi(t, 0, x) \ll_K \varphi(t, 0, y)$ for all $t \in (l_y, 0]$, which implies that $\sup_{t \in (l_y, 0]} \varphi_i(t, 0, y) < +\infty$ for all $i = k+1, \ldots, n$. Hence, $y \in \hat{R}_-(\infty) \neq \emptyset$. Now let $x \in \hat{R}_-(\infty)$ and $x <_K y$, $y \in C^o$. Suppose that $y \in \Gamma$, then $\limsup_{t \to -\infty} \varphi_i(t, 0, y) < +\infty$ for all $i \in N = \{1, \ldots, n\}$. On the other hand, there is an $i \in K = \{1, \ldots, k\}$ such that $\sup_{t \in (l_x, 0]} \varphi_i(t, 0, x) = +\infty$. For such i, $\sup_{t \in (l_x, 0]} \varphi_i(t, 0, y) = +\infty$, a contradiction. Hence, $y \notin \Gamma$. Moreover, it is also easy to see that $l_y \geqslant l_x$ for a similar reason. Then

$$\sup_{t \in (l_y, 0]} \varphi_i(t, 0, y) \leqslant \sup_{t \in (l_y, 0]} \varphi_i(t, 0, x) \leqslant \sup_{t \in (l_x, 0]} \varphi_i(t, 0, x) < +\infty,$$

for all $i \in N \setminus K$. Thus, $y \in \hat{R}_-(\infty)$, which implies that $\hat{R}_-(\infty)$ is upper closed. Similarly, we can obtain that $\hat{R}_-(\infty)$ is order convex. Now, let $x \in \hat{R}_-(\infty)$, then

$Tx \notin \Gamma$. Note that $Tx = \varphi(\tau, 0, x)$, then it follows from (1.2) that

$$\sup_{t\in(-\tau+l_x,0]} \varphi_i(t, 0, Tx) = \sup_{t\in(l_x,0]} \varphi_i(t, 0, x) < +\infty,$$

for all $i \in N \setminus K$, which implies (a). Similarly, (b) also holds.

(2) From (1), it is easy to obtain that $R_-(\infty)$ is non-empty, order convex and upper closed in C^o. Fix an $x \in R_-(\infty)$, then there is a neighborhood U of x such that $U \subset \hat{R}_-(\infty)$. By (a) in (1), $TU \subset \hat{R}_-(\infty)$. Since T is a homeomorphism, TU is a neighborhood of Tx, which implies that $Tx \in R_-(\infty)$. Furthermore, by the continuous dependence on the initial value, we can also choose U sufficiently small such that $U \in T(C)$ provided $x \in T(C)$. Then, by (b) in (1), $T^{-1}U \in \hat{R}_-(\infty)$, which implies that $T^{-1}x \in R_-(\infty)$.

(3) Fix $x \in V_-(\infty) \cap T(C)$, then there is a sequence $x_n \in R_-(\infty)$ with $x_n \gg_K x$ and $x_n \to x$. By the continuous dependence on the initial value, $T^{-1}x_n$ exists for n sufficiently large. Hence, $T^{-1}x_n \gg_K T^{-1}x$, $T^{-1}x_n \in R_-(\infty)$ and $T^{-1}x_n \to T^{-1}x$. On the other hand, we assert that there exist no $y_n \in R_-(\infty)$ such that $y_n \ll_K T^{-1}x$ and $y_n \to T^{-1}x$. If not, $[[y_n, T^{-1}x_n]]_K \subset R_-(\infty)$, which implies that $T^{-1}x \in R_-(\infty)$. Hence, $x \in R_-(\infty)$, a contradiction. Thus, we have proved $T^{-1}x \in V_-(\infty)$ if $x \in V_-(\infty) \cap T(C)$.

Let $x \in V_-(\infty)$. There is a sequence $x_l \uparrow x$, $x_l \in R_-(\infty)$. As $R_-(\infty)$ is upper closed, $\bigcup_{l=1}^{\infty}[x, +\infty)_K = [[x, +\infty)_K \subset R_-(\infty)$. Take a sequence $y_m \uparrow Tx$ such that $T^{-1}y_m$ exists. By (S), $T^{-1}y_m \gg_K x$, that is, $T^{-1}y_m \in [[x, +\infty)_K$. Hence $y_m = T(T^{-1}y_m) \subset TR_-(\infty) \subset R_-(\infty)$. Furthermore, if there exist $x_n \ll_K Tx$, $x_n \in R_-(\infty)$ with $x_n \to Tx$. Then $T^{-1}x_n$ exists for all n sufficiently large. Hence, $T^{-1}x_n \ll_K x$, $T^{-1}x_n \in R_-(\infty)$ with $T^{-1}x_n \to x$, contradicting that $x \in V_-(\infty)$. Thus, $Tx \in V_-(\infty)$.

As to the non-ordering of $V_-(\infty)$, we first show that no two points on $V_-(\infty)$ are related by $\ll_K$. Suppose that $x \ll_K y$ and $x, y \in V_-(\infty)$. Then there exist $x_n, y_n \in R_-(\infty)$, $x_n \gg_K x$, $y_n \gg_K y$ such that $x_n \to x$, $y_n \to y$. Hence, $y \in [[x_n, y_n]]_K \in R_-(\infty)$ for n sufficiently large, which contradicts $y \in V_-(\infty)$.

Next, we remark that the sets $\hat{R}_-(\infty)$, $R_-(\infty)$ and $V_-(\infty)$ can also defined with respect to the mapping $T_\theta := \varphi(\tau+\theta, \theta, \cdot)$ for each $\theta \in [0, \tau)$. We denote the resulting sets by $\hat{R}^\theta_-(\infty)$, $R^\theta_-(\infty)$ and $V^\theta_-(\infty)$, respectively. Obviously, $\hat{R}^\theta_-(\infty)$, $R^\theta_-(\infty)$ and $V^\theta_-(\infty)$ have the same properties that were proved above. Especially, we have that no two points on $V^\theta_-(\infty)$ are related by $\ll_K$. Now, suppose that $x <_K y$ and $x, y \in V_-(\infty) \subset C^o$. Then there is a small $t_0 \in (-\tau, 0)$ such that $\xi := \varphi(t_0, 0, x) \ll_K \varphi(t_0, 0, y) := \eta$. Since (1.1) is τ-periodic, we have $\varphi(t+\tau, t_0+\tau, \xi) = \varphi(t, t_0, \xi) = \varphi(t, 0, x)$. Let $\theta = t_0 + \tau \in (0, \tau)$. Then we claim that $\xi \in V^\theta_-(\infty)$. Indeed, let $x_n \downarrow x$ with $x_n \in R_-(\infty)$, then from the continuous dependence on the initial value, we can define $\xi_n := \varphi(t_0, 0, x_n)$, hence $\xi_n \downarrow \xi$. It is not difficult to see that $\xi_n \in R^\theta_-(\infty)$. Suppose that there exists a sequence $\zeta_n \uparrow \xi$ with $\zeta \in R^\theta_-(\infty)$. Then, by the order-convexity of $R^\theta_-(\infty)$, we have $\xi \in R^\theta_-(\infty)$, which implies that $x \in R_-(\infty)$, a contradiction. Thus $\xi \in V^\theta_-(\infty)$. Similarly, we obtain $\eta \in V^\theta_-(\infty)$. Since $\xi \ll_K \eta$, it contradicts the fact that $V^\theta_-(\infty)$ is unordered with respect to $\ll_K$. Thus, we have proved that no two points on $V_-(\infty)$ are related by $<_K$.

Finally, the proofs that $R_-(\infty)$ is an open n-cell, and $V_-(\infty)$ is an open $(n-1)$-cell and closed in C can be adapted from Proposition 3.3 in [18]. We omit them here. $\square$

Now, let $L \subset C^o$ be an ω- or α-set of some $v \in C^o$ which is not a cycle. We define

$$\Gamma^- = \{x \in \Gamma \cap C^o : [[x, +\infty)_K \cap \Gamma \neq \emptyset\},$$

$$\Gamma^+ = \{x \in \Gamma \cap C^o : (-\infty, x]]_K \cap \Gamma \neq \emptyset\},$$

$$L^- = \Gamma^- \cap L \qquad \text{and} \qquad L^+ = \Gamma^+ \cap L.$$

Lemma 3.4. *Suppose that $x \in L^- \neq \emptyset$. Then there is a $y \in \Gamma$ with $x \ll_K y$, such that $\alpha(z) \neq x$ for all $z \in [[x, y]_K$. Moreover, the corresponding result holds for $x \in L^+ \neq \emptyset$.*

Proof. Since $x \in L^- \neq \emptyset$, we can choose a $y \in \Gamma \cap C^o$ such that $x \ll_K y$. Note that Γ is order-convex, $[x, y]_K \subset \Gamma$. Suppose that there exists some $z \in [[x, y]_K$ such that $\alpha(z) = x$. Then, by the results contained in Mierczyński [21, Sections 1 and 2], there is a weakly invariant, totally ordered by $\ll_K$ one-dimensional submanifold W, homeomorphic to the interval $[0, 1)$, tangent at x to the principal eigenvector of $DT^{-1}(x)$, and such that either (i) $T^{-1}w \ll_K w$ for all $w \in W \backslash \{x\}$, or (ii) $T^{-1}w \gg_K w$ for all $w \in W \setminus \{x\}$, or else (iii) there is a sequence $x_k \downarrow x, x_k \in W \cap Fix(T)$. Case (iii) is impossible, since then $x_{k_0} \ll_K z$ for some k_0, and $x_{k_0} = \alpha(x) \gg_K x = \alpha(z)$. Similarly, in case (ii) there is $w \in W$ with $x \ll_K w \ll_K T^{-1}w \ll_K z$, consequently $\alpha(w) = u$ with $u \gg_K w$, which contradicts $x = \alpha(x) = \alpha(z)$. So, case (i) holds, and we can assume without loss of generality, by replacing z if necessary with any $w \in W, w \neq x,$ that $T^{-1}z \ll_K z$. Define $N = \{w \in L : w \leqslant_K z\}$. Obviously, $N \neq \emptyset$. Suppose that $N = \{x\}$, which is a fixed point. Since L is not a cycle, $L \setminus \{x\} \neq \emptyset$. Therefore, it follows from $x \ll_K z$ and the definition of N that $N = \{x\}$ is a proper, isolated and invariant set of L. On the other hand, it is known that $T|_L$ is chain transitive (see [12]) and this implies that L cannot have a proper, isolated and invariant subset. Thus we have induced a contradiction and obtained that $N \setminus \{x\} \neq \emptyset$. Fix a $w \in N \setminus \{x\}$. Since $z \in C^o$, by (A2), we can assume that $w \ll_K z$. Noticing that $w \in L$ is a non-wandering point of T^{-1}, hence there exist $n_i \to +\infty$ and $w_i \to w$ such that $T^{-n_i}w_i \to w$ as $i \to +\infty$. Then we choose $w_i \ll_K z$ for all i sufficiently large, which implies that $T^{-n_i}w_i \ll_K T^{-n_i}z$, let i tend to the infinity, we obtain that $w \leqslant_K x$. Since $w \in N \setminus \{x\}$, $w <_K x$, which contradicts Proposition 2.3. We have completed the lemma.

Lemma 3.5. *Suppose that $x \in L^- \neq \emptyset$. Then there exist two sequences $\{z_n\}$,$\{p_n\}$ in Γ satisfying $z_n \downarrow x$, $p_n \in \alpha(z_n)$ such that $p_n \downarrow p$ for some $p \geqslant_K x$. Moreover, the corresponding result holds for $x \in L^+ \neq \emptyset$.*

Proof. By Lemma 3.4, we obtain some $y \in \Gamma$ such that $\alpha(z) \neq x$ for any $z \in [[x, y]_K$. Choose some sequence $\{z_n\} \subset [[x, y]_K$ such that $z_n \downarrow x$. Since $x \in L \subset C^o$ is a non-wandering point of T^{-1}, we can choose $y_k \to x$ and $n_k \to +\infty$ such that $T^{-n_k}y_k \to x$ as $k \to +\infty$.

For $z_1 \gg_K x$, there exists a $k_0 \in \mathbb{N}$ such that $z_1 \gg_K y_k$ for any $k \geqslant k_0$. Therefore, $T^{-n_k} z_1 \gg_K T^{-n_k} y_k$ by (A2). Choose a subsequence $\{n_{1k}\}$ of $\{n_k\}$ such that $T^{-n_{1k}} z_1 \to p_1 \in \alpha(z_1)$. So $p_1 \geqslant_K x$. For $z_2 \gg_K x$, we can choose a subsequence $\{n_{2k}\}$ of $\{n_{1k}\}$ such that $T^{-n_{2k}} z_2 \to p_2 \in \alpha(z_2)$ with $p_1 \geqslant_K p_2 \geqslant_K x$. Now by the diagonal process, we obtain a sequence $\{p_k\}$ satisfying $p_1 \geqslant_K \cdots \geqslant_K p_k \geqslant_K \cdots \geqslant_K x$ and $p_k \in \alpha(z_k)$. Let $p = \lim_k p_k$. Obviously, $p \geqslant_K x$. This completes the proof. □

In the situation of Lemma 3.5, we know that $p \geqslant_K x$. We will discuss the cases

(**A**) $p = x$,

(**B**) $p >_K x$,

respectively.

We first introduce some important definitions which are due to Takáč [36,37]. A continuous mapping T is called α-compact in a subset Y of C if $O(x)$ is relatively compact for each $x \in Y$, and also $\cup_{x \in Y} \alpha(x)$ is relatively compact in C. Obviously, T is α-compact in Γ. If $x \in \Gamma^- \subset C^o$, we define the *upper α-limit set* of x as

$$\alpha_+(x) = \bigcap_{\substack{y \in \Gamma \\ y \gg_K x}} \overline{\left(\bigcup_{\substack{z \in \Gamma \\ x <_K z \leqslant_K y}} \alpha(z) \right)}.$$

Observe that $\alpha_+(x)$ is a nonempty, compact and invariant subset of Γ. Furthermore, the following three properties hold (cf. [37, Proposition 3.1 and Corollary 3.2].):

(P1) if $z_n \downarrow x$ for some sequence $z_n \in \Gamma$, then $\alpha_+(x) = \bigcap_{k=1}^\infty \overline{\left(\bigcup_{n=k}^\infty \alpha(z_n)\right)}$.

(P2) no two points of $\alpha_+(x)$ are related by $\ll_K$, and no two points of $\alpha_+(x) \cap C^o$ are related by $<_K$.

(P3) $\alpha(y) \preceq_K \alpha_-(x) \preceq_K \alpha(x) \preceq_K \alpha_+(x) \preceq_K \alpha(z)$ for all $y, z \in \Gamma$ satisfying $y <_K x <_K z$.

Similarly, we can also define the *lower α-limit set* $\alpha_-(x)$ of x if $x \in \Gamma^+ \subset C^o$. We say that x is upper (resp. lower) α-stable if $\alpha_+(x) = \alpha(x)$ (resp. $\alpha_-(x) = \alpha(x)$). Otherwise x is upper (resp. lower) α-unstable. The set of all upper (resp. lower) α-stable points is denoted by $\mathcal{S}_+(\mathcal{S}_-)$. The set of all upper (resp. lower) α-unstable points is denoted by $\mathcal{U}_+(\mathcal{U}_-)$. We also let $\mathcal{S} = \mathcal{S}_+ \cap \mathcal{S}_-$ and $\mathcal{U} = \mathcal{U}_+ \cup \mathcal{U}_-$.

Lemma 3.6. *Suppose that $x \in L^- \neq \emptyset$. Assume that case* (A) *holds. Then $x \in \alpha_+(x) = \alpha(x) \subset L \subset C^o$. Furthermore, we can choose $y \in \Gamma$ in Lemma* 3.4 *so close to x that $\alpha(z) \subset C^o$ for all $z \in [x, y]_K$.*

Proof. Since (A) holds, it follows from property (P1) that $x = p \in \alpha_+(x)$. Hence, $\alpha(x) \subset \alpha_+(x)$. Suppose that there is a $z \in \alpha_+(x) \setminus \alpha(x)$. Then, by (P3), we can find a $w \in \alpha(x) \subset L \subset C^o$ such that $w <_K z$. Hence, $T^{-1}w \ll_K T^{-1}z$, which contradicts (P2). So, we have $x \in \alpha_+(x) = \alpha(x) \subset C^o$. Now, suppose that the last statement of this lemma is false. Then there exist two sequences $\{z_n\}$ and $\{w_n\}$ such that $z_n \downarrow x$ and $w_n \in \alpha(z_n) \cap \partial C$. By the α-compactness, we assume without loss of generality that $w_n \to w \in \partial C$. On the other hand, note that $w \in \alpha_+(x) \subset C^o$, a contradiction. □

Now, we choose a compact arc J simply ordered by $\ll_K$ with endpoints x and y. By Lemma 3.6, we can choose y so close to x that $\alpha(z) \subset C^o$ for all $z \in J$ and one of the following three alternatives must occur:
(Alt$_\text{a}$) $\alpha(x) = \alpha(y)$,
(Alt$_\text{b}$) $\alpha(x) \neq \alpha(y)$ and $J \setminus \{x\} \subset \mathcal{S}$,
(Alt$_\text{c}$) $\alpha(x) \neq \alpha(y)$ and $J \setminus \{x\} \not\subset \mathcal{S}$.
The proofs of the following three lemmas are adaptations of several arguments in [42]. However, in many places here, they can be proved more simply and directly. Hence, for the reader's convenience we supply the proofs.

Lemma 3.7. *Suppose that the condition in Lemma* 3.6 *holds. Suppose also that* (Alt$_\text{a}$) *holds. Then x is a linearly stable periodic point of T^{-1}.*

Proof. Since $x \in \alpha(x) = \alpha(y)$ and $x \ll_K y$, there is some $m \in \mathbb{N}$ such that $T^{-m}y \ll_K y$. Hence, by (A2) and the convergence criterion in [37, Proposition2.1], $\alpha(y)$ is a cycle. Thus, x is an m-periodic point. Assume without loss of generality that m is the prime period of x. Suppose that x is not linearly stable with respect to T^{-1}. Then $\rho(DT^{-m}(x)) > 1$. By Theorem 2.1 in [30], there exists a unique invariant curve $\beta(s)$ satisfying $\beta(0) = x, \beta(s_1) \ll_K \beta(s_2)$ for $0 \leqslant s_1 < s_2$. Since $x \in L^-$, $\beta(s)$ is well defined on $[0, +\infty)$ and $\lim_{s\to+\infty} \beta(s) = x_2$ where $T^m x_2 = x_2$ and $T^{-mk}z \to x_2$ as $k \to +\infty$ for all $z \in [x, x_2]_K \setminus \{x\}$. Then, obviously, $\alpha(z) = O(x_2)$ for all $z \in [x, x_2]_K \setminus \{x\}$. In particular, we choose $l \in \mathbb{N}$ sufficiently large such that $T^{-l}y \in [x, x_2]_K$, which implies that $O(x) = \alpha(x) = \alpha(y) = \alpha(T^{-l}y) = O(x_2)$, a contradiction. $\square$

Lemma 3.8. *Suppose that the condition in Lemma* 3.6 *holds. Suppose also that* (Alt$_\text{b}$) *holds. Then the following statements hold true*:

(1) *every strictly increasing sequence in $\mathbb{N}$ contains a subsequence $n_1 < n_2 < \cdots$ such that for every $z \in J$ we have*

$$T^{-n_k}z \to \phi \text{ as } n_k \to +\infty, \textit{ for some } \phi = \phi(z).$$

Furthermore, $\phi : J \to \Gamma$ is increasing (*i.e., $x_1 \leqslant_K x_2$ implies $\phi(x_1) \leqslant_K \phi(x_2)$*) *and continuous. In particular, the image of ϕ is either a simply ordered, closed arc or else a singleton.*

(2) *x is a linearly stable periodic point of T^{-1}.*

Proof. Statement (1) is essentially due to Takáč and can be adapted from [37]. We focus on statement (2). First note that $x \in \alpha(x) \subset C^o$, then it is easy to see that $x \notin \alpha(y)$ by (Alt$_\text{b}$) (Otherwise, $x \in \alpha(y) \cap \alpha(x) \subset C^o$. By Proposition 2.4, $\alpha(x) \preceq_K \alpha(y)$, and hence there exists $z \in \alpha(y)$ such that $x <_K z$. Note that $x, z \in \alpha(y)$, a contradiction to Proposition 2.3). Now, choose a sequence $n_k \to +\infty$ such that $T^{-n_k}x \to x$ and $T^{-n_k}y \to w$ as $k \to +\infty$. Hence, $x <_K w$. Then, by (A2), we can assume that $x \ll_K w$ without loss of generality. Choose some positive integer m such that $T^{-m}x \leqslant_K w$

and define

$$\mathcal{P} = \{z \in J : \{x, T^{-m}x\} \preceq_K w \text{ for some } w \in \alpha(z)\}.$$

Obviously, $\mathcal{P} \neq \emptyset (y \in \mathcal{P})$. Let $z_0 = \inf \mathcal{P}$. Then there exists $\{z_n\} \subset J, z_n \downarrow z_0 \in J$ and $\{x, T^{-m}x\} \preceq_K w_n$ for some $w_n \in \alpha(z_n)$. By α-compactness of T in Γ we can assume that $w_n \to w_0$. Then $w_0 \in \alpha_+(z_0)$. Note that $z_0 \in J$ and (Alt$_\mathrm{b}$), we have $w_0 \in \alpha(z_0)$ and

$$\{x, , T^{-m}x\} \preceq_K w_0. \tag{3.1}$$

Then $z_0 \in \mathcal{P}$. From (3.1), we have either

$$\text{(a)} \quad x = w_0 \quad \text{or} \quad T^{-m}x = w_0,$$

or else

$$\text{(b)} \quad x <_K w_0 \quad \text{and} \quad T^{-m}x <_K w_0.$$

Suppose that (a) holds. Then $T^{-m}x \leqslant_K x$ or $T^{-m}x \geqslant_K x$. Hence $\alpha(x)$ is a cycle, which implies that x is a periodic point.

Suppose that (b) holds. We claim that $z_0 = x$. Suppose that $z_0 \gg_K x$. From (b) and (A2), we obtain $T^{-1}x \ll_K T^{-1}w_0$ and $T^{-m-1}x \ll_K T^{-1}w_0$. Since $T^{-1}w_0 \in \alpha(z_0)$, there exists a sequence $n_k \to +\infty$ such that $T^{-n_k}z_0 \to T^{-1}w_0$ as $n_k \to +\infty$. Then, by statement (1) in this Lemma, we can choose a subsequence, still denoted by $\{n_k\}$, such that $T^{-n_k}z \to \phi(z)$ for all $z \in J$. Obviously, $\phi(z_0) = T^{-1}w_0$. Because ϕ is continuous in J, then there is some $z' \in J$ satisfying $z' \ll_K z_0$ such that $T^{-1}x \ll_K w'$ and $T^{-m-1}x \ll_K w'$, where $w' = \phi(z') \in \alpha(z')$. Hence $T^{-n}x \ll_K T^{-n+1}w'$ and $T^{-m-n}x \ll_K T^{-n+1}w'$ for all $n \geqslant 1$. Now choose a sequence $\{n_l\}$ such that $T^{-n_l}x \to x$ and $T^{-n_l+1}w' \to w'' \in \alpha(z')$. Therefore, $x \leqslant_K w'', T^{-m}x \leqslant_K w''$ and $w'' \in \alpha(z')$. Because $z' \ll_K z_0$, this contradicts the minimal property of z_0. We have now proved the claim that $z_0 = x$. Now, by (b), we have $x <_K w_0$ and $T^{-m}x <_K w_0$ for some $w_0 \in \alpha(x)$. By Proposition 2.3, we obtain that $x = w_0 = T^{-m}x$, which implies that x is an m-periodic point.

Finally, we prove that x is a linearly stable m-periodic point with respect to T^{-1}. If not, then $\rho(\lambda_1(x)) > 1$. By Theorem 2.1 in [30], there exists a unique invariant curve $y(s)$ satisfying $y(0) = x, y(s_1) \ll_K y(s_2)$ for $0 \leqslant s_1 < s_2$. Since $x \in L^-$, $y(s)$ is well defined on $[0, +\infty)$ and $\lim_{s \to +\infty} y(s) = x_2$ where $T^m x_2 = x_2$ and $T^{-nm}y \to x_2$ as $n \to +\infty$ for all $y \in [x, x_2]_K \setminus \{x\}$. Then, obviously, $\alpha(y) = O(x_2)$ for all $y \in [x, x_2]_K \setminus \{x\}$. Thus $\alpha_+(x) = O(x_2)$. Hence, $\alpha(x) = O(x) \neq O(x_2) = \alpha_+(x)$, contradicting Lemma 3.6. $\square$

Lemma 3.9. *Suppose that the condition in Lemma* 3.6 *holds. Suppose also that* (Alt$_\mathrm{c}$) *holds. Then x is a linearly stable periodic point of T^{-1}.*

Proof. Since ($\mathrm{Alt_c}$) holds, there exists a sequence $z_n \in J \cap \mathcal{U}$ such that $z_n \downarrow x$. Assume that $z_n \in \mathcal{U}_-$ without loss of generality. Then we have the following:

Claim. *For each z_n, there exists a $y_n \in J$ satisfying $y_n \ll_K z_n$ such that $\alpha(y) = \alpha_-(z_n)$ for all $y \in J \cap [y_n, z_n]]_K$. Furthermore, $\alpha_-(z_n)$ is a linearly stable cycle of T^{-1}.*

Before giving the proof we show how this claim implies Lemma 3.9. Let $y_n \in J$, $y_n \downarrow x$ and $\alpha(y_n)$ a k_n-cycle be as in the claim. Then for any $z \in \alpha(x)$, there exist

$$q_n \in \alpha(y_n) \text{ such that } q_n \downarrow z.$$

Indeed, by Proposition 2.4, we have $\alpha(x) \preceq_K \alpha(y_n)$. Then for any $z \in \alpha(x)$, we choose $q_n \in \alpha(y_n)$ such that $z \leqslant_K q_n$. Furthermore, by Proposition 2.4, $\alpha(y_{n+1}) \preceq_K \alpha(y_n)$ we can choose q_n decreasing with respect to "$\leqslant_K$". Note, by Lemma 3.6, that $\alpha(y_n) \subset C^o$ for all $n \in \mathbb{N}$. Then, by passing to a subsequence if necessary, we can further assume that $\{q_n\}$ decreasing with respect to "$\ll_K$". (Otherwise, $q_n = q_{n+1}$ for all n sufficiently large. Since $\alpha(y_n)$ are cycles and $q_n \in \alpha(y_n)$, then $\alpha(y_n) = \alpha(y_{n+1})$ for all n sufficiently large. Note that $y_{n+1} \ll_K z_{n+1} \ll_K y_n$, then $\alpha(y_n) = \alpha(z) = \alpha(y_{n+1})$, which contradicts $z \in \mathcal{U}_-$.) Thus we can assume that $q_n \downarrow \tilde{z} \in \alpha_+(x) = \alpha(x)$. By Proposition 2.3, we have $z = \tilde{z}$, thus $q_n \downarrow z$.

Since assumption (A) holds, by property (P1) and Lemma 3.6, we obtain that $\alpha(x) = \alpha_+(x) = \cap_{k \geqslant 1} \overline{(\cup_{n \geqslant k} O(q_n))}$, where $O(q_n)$ is a linearly stable k_n-cycle of T^{-1} in C^o. Note that $\alpha(x) \subset C^o$ and T satisfies the fundamental assumption (A2). By Lemma 4.1, followed by some important arguments, in Hess and Poláčik [6], we conclude that $\alpha(x)$ contains a linearly stable cycle $O(z)$ of T^{-1}, where $z \in \alpha(x)$. Also by Lemma 4.2 and Proposition 4.3 in [6], we have $\liminf_n k_n = k < +\infty$, where k_n is the period of $O(q_n)$. That is to say, there exists a subsequence $\{k_{n_j}\} \subset \{k_n\}$ such that $\lim_j k_{n_j} = k$. Since $k_{n_j} \in \mathbb{N}$, we obtain that $k_{n_j} \equiv k$ for j sufficiently large. Note that $x \in \alpha(x)$, then x is a k-periodic point and $O(x) = \alpha(x) = O(z)$ is a linearly stable k-cycle of T^{-1} in C^o. So, it remains to prove the claim.

Proof of the Claim. It follows from Proposition 2.1 and (P2) for any fixed n, there exists an invariant order decomposition (A, B) of C such that $\alpha_-(z_n) \subset H = A \cap B$. Since $\alpha_-(z_n) \preceq_K \alpha(z_n)$, one has $\alpha(z_n) \subset B$. We assert that $z_n \in \mathrm{Int}\, B$. Suppose that $z_n \in A$. Then $\alpha(z_n) \subset A$, which implies that $\alpha(z_n) \subset H = A \cap B$. Since $z_n \in \mathcal{U}_-$, there are some $y \in \alpha(z_n)$ and some $v \in \alpha_-(z_n)$ such that $v <_K y$. It follows from Lemma 3.6 that $\alpha(z_n) \subset C^o$. Hence, we can assume that $v \ll_K y$, which contradicts the fact that H is unordered with respect to $\ll_K$.

Now we obtain that $z_n \in \mathrm{Int}\, B$. Then we can choose y_n so close to z_n that $y_n \in (J \cap \mathrm{Int}\, B) \setminus \{x\}$. Then $\alpha(y) \subset B$ for any $y \in [y_n, z_n]]_K$. On the other hand, for such y, $\alpha(y) \preceq_K \alpha_-(z_n)$ by (P3), then $\alpha(y) \subset A$. Hence, $\alpha(y) \subset H = A \cap B$ for all $y \in [y_n, z_n]]_K$. Note that $\alpha_-(z_n) \subset H$ and $\alpha_-(z_n) \subset C^o$, then it is easy to see that $\alpha(y) = \alpha_-(z_n)$ for all $y \in [y_n, z_n]]_K$, which is exactly the first statement of the Claim.

Finally, we shall show that $\alpha_-(z_n)$ is a linearly stable cycle of T^{-1}. Since $\alpha_-(z_n) \neq \alpha(z_n)$ and $\alpha_-(z_n), \alpha(z_n) \subset C^o$, there exist $p \in \alpha_-(z_n) \setminus \alpha(z_n)$ and $q \in \alpha(z_n)$ such that $p \ll_K q$.

Now, we define $Y_0 = \{y \in C^o : p <_K y <_K q\}$. Obviously, $Y_0 \neq \emptyset$. Let $y \in Y_0$, then, by Proposition 2.4, $\alpha(p) \preceq_K \alpha(y) \preceq_K \alpha(q)$, $\alpha(p) \subset \alpha_-(z_n) \subset C^o$ and $\alpha(q) \subset \alpha(z_n) \subset C^o$. By Proposition 2.1, we can find an invariant decomposition (A, B) of C such that $\alpha(y) \subset H = A \cap B$. Then

$$\alpha(p) \subset A \quad \text{and} \quad \alpha(q) \subset B. \tag{3.2}$$

Suppose that $z_n \in A$. Then $\alpha(q) \subset \alpha(z_n) \subset A$. Thus $\alpha(q) \subset H = A \cap B$. Hence, $\alpha(y) = \alpha(q)$ by the non-ordering of H. Therefore,

$$\alpha(y) = \alpha(q) \subset \alpha(z_n). \tag{3.3}$$

From (A2), we have $T^{-1}p \ll_K T^{-1}y \ll_K T^{-1}q$. Now we choose some $m \in \mathbb{N}$ such that $T^{-1}y \ll_K T^{-m}z_n$. Hence, by the continuity of T^{-1}, there exists some $a^* \ll_K z_n$ such that $T^{-1}y \ll_K T^{-m}w$ for all $w \in [a^*, z_n]]_K$. Then $\alpha(y) \preceq_K \alpha(w) \preceq_K \alpha(z_n)$ for all $w \in [a^*, z_n]]_K$. By (3.3), we obtain that $\alpha(w) = \alpha(y) = \alpha(z_n)$ for all $w \in [a^*, z_n]]_K$. By the definition of $\alpha_-(z_n)$, we have $\alpha_-(z_n) = \alpha(w) = \alpha(z_n)$. But $p \in \alpha_-(z_n) \setminus \alpha(z_n)$, a contradiction. Thus, $z_n \in \operatorname{Int} B$.

Now, we can find some $\bar{z} \ll_K z_n$ such that $\bar{z} \in B \cap J$. Let $v = \max\{y_n, \bar{z}\}$, where y_n is obtained in the Claim's first statement. Then we have

$$\alpha(u) = \alpha_-(z_n) \subset B \text{ for any } u \in [v, z_n]]_K. \tag{3.4}$$

Thus $\alpha(p) \subset \alpha_-(z_n) \subset B$. By (3.2), we obtain $\alpha(p) \subset H = A \cap B$. Hence $\alpha(p), \alpha(y) \subset H$, $\alpha(p) \preceq_K \alpha(y)$ and $\alpha(p) \subset C^o$. Then we can prove that $\alpha(p) = \alpha(y)$, which implies that

$$\alpha(y) \subset \alpha_-(z_n). \tag{3.5}$$

Since $T^{-1}p \in \alpha_-(z_n)$, by property (P1), we choose $v_k \in J, v_k \uparrow z_n$ and $p_k \in \alpha(v_k)$ such that $p_k \to T^{-1}p$ as $k \to +\infty$. Then we can find some $k_0 \in \mathbb{N}$ such that $p_{k_0} \ll_K T^{-1}y$ and $v \ll_K v_{k_0} \ll_K z_n$. We can also choose some $m_0 \in \mathbb{N}$ such that $T^{-m_0}v_{k_0} \ll_K T^{-1}y$. Then, by (3.4) and Proposition 2.4,

$$\alpha_-(z_n) = \alpha(v_{k_0}) \preceq_K \alpha(y). \tag{3.6}$$

It follows from (3.5) and (3.6) that $\alpha(y) = \alpha_-(z_n)$. Notice that $p \in \alpha_-(z_n)$, then $p \in \alpha(y)$. Since $p <_K y <_K q$, by copying the reasoning in the proof of Lemma 3.7, $\alpha(y)$ is a linearly stable periodic point of T^{-1}. That is, $\alpha_-(z_n)$ is a linearly stable cycle of T^{-1}. This completes the proof. $\square$

Proposition 3.10. *Suppose that* $x \in L^- \neq \emptyset$*. Then there exist a periodic point* p *with* $L \prec\prec_K O(p)$ *and a sequence* $z_n \downarrow x$ *such that* $\alpha(z_n) = O(p)$ *for all* n *sufficiently large. Moreover, the corresponding result holds for* $x \in L^+ \neq \emptyset$*.*

Proof. Since $x \in L^- \neq \emptyset$, it follows from Lemma 3.5 that there exist two sequences $\{z_n\}, \{p_n\}$ in Γ satisfying $z_n \downarrow x$, $p_n \in \alpha(z_n)$ such that $p_n \downarrow p$ and $p \geqslant_K x$.

We claim that only case (B), i.e., $p >_K x$, holds. Indeed, suppose that $p = x$. Then, by Lemmas 3.7–3.9, we obtain that x is a positive and linearly stable m-periodic point of T^{-1} and $\alpha(x) = O(x)$. That is, $\rho(DT^{-m}(x)) \leqslant 1$. It follows from Proposition 3.5 in [23] that there is a neighborhood $\mathcal{V}$ of x such that for any $y \in \mathcal{V}$, either $T^{-mn}y \to x$ as $n \to +\infty$ or there is some $n_0 \geqslant 0$ such that $T^{-mn_0}y$ and $T^{-m}x(= x)$ are related by the order $\leqslant_K$. Since L is not a cycle, we obtain that $(\mathcal{V} \cap L) \setminus \{x\} \neq \emptyset$. Then for any $y \in (\mathcal{V} \cap L) \setminus \{x\}$, by Proposition 2.3, we have $T^{-mn}y \to x$ as $n \to +\infty$. Then $x \in \mathcal{V} \cap L$ is a proper attractor of T^{-m} in L. On the other hand, it is known that L is a chain transitive set of T^{-m}(cf. [12]), a contradiction. We have proved the claim.

From (A2) we have $T^{-1}p \gg_K T^{-1}x$. Hence, there is some $l_0 \in \mathbb{N}$ such that $T^{-1}z_l \ll_K T^{-1}p$ for all $l \geqslant l_0$. Thus we get

$$T^{-1}z_l \ll_K T^{-1}p <_K T^{-1}p_l \quad \text{for all } l \geqslant l_0. \tag{3.7}$$

Note that $p_l \in \alpha(z_l)$, then by the convergence criterion in [37, Proposition 2.1], $\alpha(z_l)$ is a cycle and p_l is a periodic point. Furthermore, by (3.7), we obtain that $\alpha(z_l) = \alpha(p)$ for all $l \geqslant l_0$. Suppose that $\alpha(p)$ is an m-cycle. Then $p_l (l \geqslant l_0)$ is an m-periodic point, which implies that p is also an m-periodic point and $p = p_l$ for all $l \geqslant l_0$. Thus we can assume that

$$\alpha(z_l) = O(p) \text{ for all } l \geqslant 0. \tag{3.8}$$

Without loss of generality, we can assume that $L = \omega(v)$ for some $v \in C^o$ and $x \ll_K p$. Then there exist $n_k \to +\infty$ such that $T^{n_k}v \ll_K p$ for n_k sufficiently large. Given any $y \in L$. Then we can find $l_i \to +\infty$ such that $T^{l_i}v \to y$ as $i \to +\infty$. For each l_i, we choose $n_{k_i} \in \{n_k\}$ such that $n_{k_i} \geqslant l_i$ and $n_{k_i} = l_i + \lambda(l_i)$, where $\lambda(l_i) \geqslant 0$. Thus $T^{l_i}v = T^{-\lambda(l_i)}(T^{n_{k_i}}v) \ll_K T^{-\lambda(l_i)}p$ for i sufficiently large. Furthermore, we have $-\lambda(l_i) = m\tau(l_i) + \eta(l_i)$, where $0 \leqslant \eta(l_i) \leqslant m-1$. Then $T^{l_i}v \ll_K T^{\eta(l_i)}(T^{m\tau(l_i)}p) = T^{\eta(l_i)}p$ for i sufficiently large. Let $i \to +\infty$, and assume that $\eta(l_i) \to j \in \{0, 1, \ldots, m-1\}$, then $y \leqslant_K T^j p$. Since $x \ll_K p$, $p \notin L$. Then $y <_K T^j p$. Therefore, by the arbitrariness of y, we conclude that $L \prec_K O(p)$. Note that $L \subset C^o$, then we have $L \prec\prec_K O(p)$ by (A2).

The proof of the case $x \in L^+ \neq \emptyset$ is similar. $\square$

From Proposition 3.10, we obtain that for any $x \in L^- \neq \emptyset$, there exists a periodic point p_x such that $L \prec\prec_K O(p_x)$. Define

$$\mathcal{G} = \{p_x : x \in L^-, L \prec\prec_K O(p_x) \text{ and } O(p_x) \text{ is a cycle}\}.$$

Lemma 3.11. *Suppose that $L^- \neq \emptyset$. Then $\mathcal{G}$ endowed with the "$\leqslant_K$" ordering possesses a minimal element. The corresponding result holds for $L^+ \neq \emptyset$.*

Proof. By Zorn's Lemma, it is sufficient to prove: For any sequence $\{p_n\} \subset \mathcal{G}$ which is totally ordered, i.e., $p_1 \geqslant_K p_2 \geqslant_k \cdots \geqslant_K p_n \geqslant_K \cdots$, $\{p_n\}$ has a lower bound in $\mathcal{G}$. To end this, noticing that $\mathcal{G} \subset \Gamma$, let $p_n \to p$ as $n \to +\infty$. We shall prove that $p \in \mathcal{G}$.

First, it follows from the compactness of L that there is a $w \in L \subset C^o$ such that $w \leqslant_K p$. In Then we can find an index set J satisfying $\{1, 2, \ldots, k\} \subset J \subset N$ such that $p \in \dot{H}_J^+$. Since $p_n \geqslant_K p$, $p_n \in H_J^+$. Then $p_n \in \dot{H}_J^+$ for all n sufficiently large. Hence, we can assume that $p_n \in \dot{H}_J^+$ for all $n \geqslant 1$, which implies that $O(p_n) \subset \dot{H}_J^+$ for all $n \geqslant 1$. Now we restrict system (1.1)–(1.2) on the invariant face H_J^+. It is easy to see that T also satisfies assumptions (A1)–(A4) on H_J^+. We will still use the same notations as in C.

Let $M = \cap_{k \geqslant 1} \overline{(\cup_{n \geqslant k} O(p_n))}$. It is easy to see that M is a compact, invariant set in H_J^+ unordered with $\ll_K$. Furthermore, $p \in M \subset \alpha_+(p)$. We assert that $p \in \alpha(p)$. If not, then it follows from $\alpha(p) \preceq_K \alpha_+(p)$ that there is a $z \in \alpha(p)$ such that $z <_K p$. Note that $p \in \dot{H}_J^+$, then $T^{-1}z \ll_K T^{-1}p$. Since $p \in \alpha_+(p)$, we have $\alpha(p) \subset \alpha_+(p)$. So $\alpha_+(p)$ contains two points related by $\ll_K$, a contradiction. Therefore, $p \in \alpha(p) \subset M \subset \alpha_+(p)$.

Now, we claim that $\alpha(p) \subset \dot{H}_J^+$. Indeed, note that $\alpha(p) \preceq_K \alpha(p_n) = O(p_n)$ and $O(p_n) \subset \dot{H}_J^+$, then $(\alpha(p))_j > 0$ for all $j \in J \setminus \{1, 2, \ldots, k\}$. Suppose that $\alpha(p) \not\subset \dot{H}_J^+$, then there exists $I \subset \{1, 2, \ldots, k\}$ such that $Q := \alpha(p) \cap H^+_{(J \setminus \{1,2,\ldots,k\}) \cup I} \neq \emptyset$. Let

$$Q^+ = \{y \in \Gamma : y \geqslant_K z \text{ for some } z \in Q\}.$$

Then it is easy to see that Q^+ is a compact non-empty set ($Q \subset Q^+$) which is positively invariant to the mapping T^{-1}. Hence, it follows from the positive invariance of Q^+ that $O(q_n)(= O(p_n))$ are contained in Q^+. Hence $M \subset Q^+$. Note that $p \in M$, then there is a $z \in Q$ such that $p \geqslant_K z$. Since $p \in \dot{H}_J^+$, we have $p >_K z$ which implies that $T^{-1}p \gg_K T^{-1}z$. Thus, $\alpha(p)$ contains two pints related by $\ll_K$, a contradiction to Proposition 2.3. We have proved the claim.

We have obtain that $p \in \alpha(p) \subset M \subset \alpha_+(p)$ and $\alpha(p) \subset \dot{H}_J^+$. Then we can use the similar method in Lemma 3.6 to get $p \in \alpha(p) = M = \alpha_+(p) \subset \dot{H}_J^+$. Suppose on the contrary that $O(p)$ is not a linearly stable cycle of T^{-1}. As $\alpha(p) \subset \dot{H}_J^+$, an application of Theorem 4.1 in [23] to the restriction of system (1.1) to $\dot{H}_J^+$ gives the existence of $\delta > 0$ such that $\limsup_{l \to \infty} \left\| T^{-l}v - T^{-l}y \right\| \geqslant \delta$ for any $v \geqslant_K p$ with $v \in \dot{H}_J^+$. Since $p \in \alpha(p)$, there is a sequence $\{l_m\}$ such that $T^{-l_m}p \to p$ as $m \to \infty$. Fix for the moment p_n. By passing to a subsequence, if necessary, we can assume that $T^{-l_m}p_n \to r_n \in O(p_n)$. We have $\|p - r_n\| \geq \delta$ and $r_n \geqslant_K p$. Take a subsequence $\{r_{n_j}\}$ such that $r_{n_j} \to r$ as $j \to \infty$. Observe that $\|p - r\| \geqslant \delta$ and $r \geqslant_K p$, which yields $r >_K p$. As both p and r are in $M = \alpha(p)$, this contradicts Proposition 2.3 applied to system (1.1) restricted to $\dot{H}_J^+$. (Of course, in order for Theorem 4.1 in [23] to work one has to assume that $p_n >_K p$ for all n. But this can be done by taking a subsequence of $\{p_n\}$). Thus p is a linearly stable k-periodic point and $\alpha(p) = O(p)$.

It remain to prove $L \prec\prec_K O(p)$. On the one hand, given any $x \in L$, there exist $q_n \in O(p_n)$ such that $x \leqslant_K q_n$. Let $q_n \to q \in M = \alpha(p) = O(p)$, then $x \leqslant_K q \in O(p)$. Suppose that $x = q$, then $O(p) \subset L$. Since L is not a cycle, we can choose a $y \in L \setminus O(p)$, we can do as above to obtain a $q' \in O(p)$ such that $y <_K q'$, contradicting Proposition 2.3. Hence, we have $x <_K q$. Note that $L \subset C^o$, we can get $x \ll_K q$ without loss of generality. On the other hand, for any $q \in O(p)$, there are $q_n \in O(p_n)$ such that $q_n \downarrow q$. Since $L \prec\prec_K O(p_n)$, there exist $z_n \in L$ such that $z_n \ll_K q_n$. We can assume that $z_n \to z \in L$. So $z \leqslant_K q$. For the same reason above, we obtain $z \ll_K q$. We have completed the proof. □

Proposition 3.12. *Suppose that $L^- \neq \emptyset$. Then there exists a periodic point p such that $L \subset V_-(O(p))$. The corresponding result holds if $L^+ \neq \emptyset$.*

Proof. We only prove the case $L^- \neq \emptyset$. For any $x \in L^-$, by Proposition 3.10, there exist a cycle $O(p_x)$ with $L \prec\prec_K O(p_x)$ and a sequence $z_n \downarrow x$ such that (3.8), i.e.,

$$\alpha(z_n) = O(p_x) \text{ for all } n \geqslant 0, \tag{3.9}$$

holds. By Lemma 3.11, choose p being a minimal element of $\mathcal{G}$. Then, $L \prec\prec_K O(p)$. Therefore, there is a $q \in O(p)$ such that $x \ll_K q$, and hence, $z_n \ll_K q$ for all n sufficiently large. By (3.9) and Proposition 2.4, we have $O(p_x) = \alpha(z_n) \preceq_K \alpha(q) = O(p)$. So, for p, we can find a $q_x \in O(p_x)$ with $q_x \leqslant_K p$ such that $O(q_x) = O(p_x)$ is a cycle and $L \prec\prec_K O(q_x)$. Then it follows from the minimality of p in $\mathcal{G}$ that $q_x = p$, and hence, $O(p) = O(q_x) = O(p_x)$. Therefore, $\alpha(z_n) = O(p)$ for all $n \geqslant 0$.

To show $x \in V_-(O(p))$, it suffices to prove that there is no sequence $\{y_l\}$ with $y_l \uparrow x$ and $\alpha(y_l) = O(p)$. Indeed, if such $\{y_l\}$ exists, then $\alpha(x) = O(p) \subset L$. However, $L \prec\prec_K O(p)$, a contradiction to Proposition 2.3. By the arbitrariness of x, we obtain $L \subset V_-(O(p))$, which completes the proof. □

Now, we can obtain the following classification Theorem

Theorem 3.13. *Let $L \subset C^o$ be an α- or ω-limit set which is not a cycle. Then one of the following alternatives must occur*:

(i) *there are $p, q \in \mathrm{Per}(T)$ such that $L \subset V_-(O(p)) \cap V_+(O(q))$*;

(ii) *there is a $p \in \mathrm{Per}(T)$ such that $L \subset V_-(O(p)) \cap V_+(\infty)$*;

(iii) *there is a $q \in \mathrm{Per}(T)$ such that $L \subset V_-(\infty) \cap V_+(O(p))$*;

(iv) $L \subset V_-(\infty) \cap V_+(\infty)$.

Proof. By Proposition 3.12, it is sufficient to consider the case $L^- = \emptyset$. Similar result holds for $L^+ = \emptyset$. Suppose that $L^- = \emptyset$, then we claim that $L \subset V_-(\infty)$. Indeed, given any $y \in L$. Choose a sequence $z_n \gg_K y$ with $z_n \to y$. Since $L^- = \emptyset$, then $z_n \in R(\infty) \cap C^o$. Since $z_n \gg_K y$, it follows that $\sup_{t\in(l_{z_n},0]} \varphi_i(t, 0, z_n) < +\infty$ for all $i \in N \setminus K$, which implies that $z_n \in \hat{R}_-(\infty)$. It is also easy to see that $z_n \in R_-(\infty)$. Hence, $y \in V_-(\infty)$. To summarize: $L^- \cap L^+ \neq \emptyset$ implies (i); $L^- \neq \emptyset$ and $L^+ = \emptyset$ implies (ii); $L^+ \neq \emptyset$ and $L^- = \emptyset$ implies (iii); and $L^- \cup L^+ = \emptyset$ implies (iv). $\square$

Now we define the family $\mathcal{F} = \{M_0, M_1\} \bigcup \{M_i : i \in \Lambda\}$ of sets , where $M_i = V_-(O(p)) \neq \emptyset$ for some m-cycle $O(p)$, $M_0 = V_-(\infty)$ and $M_1 = V_+(\infty)$. By Proposition 3.2, the elements of $\mathcal{F}$ are pairwise disjoint in C. It is also easy to see that the index set Λ is at most countable. Therefore, we denote $\mathcal{F} = \{M_0, M_1; M_2, M_3 \cdots\}$.

Proposition 3.14. *Let M_i be an element of $\mathcal{F}$. Suppose $x \in C^o \setminus M_i$ is such that $\omega(x) \subset M_i \cap C^o$. Then there exists some $l \in \mathbb{N}$ such that for such l, there is a unique $y_l \in H \cap C^o$ with $\|T^{n+l}x - T^n y_l\| \to 0$ as $n \to \infty$ and either $T^{n+l}x \ll_K T^n y_l$ or $T^{n+l}x \gg_K T^n y_l$ for all $n \in \mathbb{N}$. In particular, if $i \geqslant 2$, then $l = 0$.*

Proof. Let us first consider the existence of $l \in \mathbb{N}$ and $y_l \in M_i \cap C^o$. For the cases $i \geqslant 2$. Choose $z \in \omega(x) \subset M_i \cap C^o$. Then, by Proposition 2.2, there exists a neighborhood $\mathcal{N}$ such that every point in $\mathcal{N} \setminus M_i$ is related by $\ll_K$ or $\gg_K$ to some point in M_i. Thus, by the invariance of $M_i (i \geqslant 2)$ which was obtained in Proposition 3.2, we can repeat the proof in [41, Proposition 3.6] step by step to get $l = 0 \in \mathbb{N}$ and $y_l \in H \cap C^o$ respectively.

For the cases $i = 0, 1$. Note that M_0 or M_1 is only weakly invariant according to Proposition 3.3. We have to prove it separately. Let $i = 0$. The $M_0 = V_-(\infty)$. The case $i = 1$ is similar. For $n \in \mathbb{N}$, we define $A_+(T^n x) = [T^n x, +\infty)_K \cap V_-(\infty)$ and $A_-(T^n x) = (-\infty, T^n x]_K \cap V_-(\infty)$. It is easy to see that $A_+(T^n x) \cap A_-(T^n x) = \emptyset$. Since $\omega(x) \subset C^o \cap V_-(\infty)$, it follows from Proposition 2.2 that either $A_+(T^n x)$ or $A_-(T^n x)$ is not empty for all n sufficiently large. By the continuous dependence on the initial value, there is a compact neighborhood $\mathcal{V}$ of $\omega(x)$ such that $T^{-1}y$ exists for every $y \in \mathcal{V}$. We can also choose V smaller so that there is a positive integer l such that $\bigcup_{n \geqslant l} \{A_+(T^n x) \cup A_-(T^n x)\} \subset \mathcal{V}$. Otherwise, we can find another compact neighborhood $\mathcal{W}$ of $\omega(x)$ with $\mathcal{V} \subset \mathcal{W}$ and two sequences $n_k \to +\infty$, $y_{n_k} \in A_+(T^{n_k}x) \cup A_-(T^{n_k}x)$ such that $y_{n_k} \in (\mathcal{W} \setminus \mathcal{V}) \cap V_-(\infty)$. Without loss of generality, we can assume that $y_{n_k} \gg_K T^{n_k}x (k \geqslant 0)$, $y_{n_k} \to y \in V_-(\infty)$ and $T^{n_k}x \to v \in \omega(x) \subset V_-(\infty) \cap \operatorname{Int} \mathcal{V}$. Therefore, $y >_K v$, which contradicts Proposition 3.3(3).

Now suppose that $A_+(T^l x) \neq \emptyset$. Then we claim that $A_+(T^{n+l}x) \neq \emptyset$ for all $n \geqslant 0$. Suppose that there is an $m \geqslant 1$ such that $A_+(T^{n+l}x) \neq \emptyset$ for $0 \leq n \leqslant m$ but $A_-(T^{m+1+l}x) \neq \emptyset$. Then we can find a $y \in V_-(\infty)$ such that $T^{m+1+l}x \gg_K y$. Since $y \in \mathcal{V}$, $T^{-1}y$ exists and hence $T^{m+l}x \gg_K T^{-1}y$, contradicting $A_+(T^{m+l}x) \neq \emptyset$. We have proved the claim.

Indeed, from the proof in the previous paragraph, we can also obtain that $T^{-n}A_+(T^{n+l}x) \neq \emptyset$ for all $n \geqslant 0$. Now, define $A = \bigcap_{n \geqslant 0} T^{-n}A_+(T^{l+n}x)$ which is not an empty set. Choose $y_l \in A$, then $T^n y_l \geqslant_K T^{l+n}x$ for all $n \geqslant 0$. Since $x \notin V_-(\infty)$, $T^n y_l >_K T^{l+n}x$ for all $n \in \mathbb{N}$. Since T satisfies Assumption (A2), $T^n y_l \gg_K T^{l+n}x$ for all $n \in \mathbb{N}$. Suppose that $\|T^{l+n}x - T^n y_l\| \not\to 0$ as $n \to \infty$. Then we can choose a subsequence n_i such that $T^{l+n_i}x \to a \in \omega(x) \subset V_-(\infty) \cap C^o$ and $T^{n_i}y_l \to b \in V_-(\infty)$. Then $a <_K b$, which contradicts Proposition 3.3(3).

In preparation for the proof of the uniqueness of y_l, some simple notations will be first introduced. Let V be the n-dimensional real vector space. Let $M \subset \mathbb{R}^n$, the tangent bundle TM of M is $M \times V$. The type-K cone P, with nonempty interior P^o, in V is the set of all v in V such that $v_i \geqslant 0$ for all $i \in \{1, 2, \dots, k\}$ and $v_j \leqslant 0$ for all $j \in N \setminus \{1, 2, \dots, k\}$.

Now we turn to prove the uniqueness of y_l. Without loss of generality, we set $l = 0$. Since $\omega(x) \subset C^o$ and T is strongly type-K competitive in C^o, we have the following lemma:

Lemma 3.15. *Let $\omega(x) \subset C^o$. Then there exists a compact neighborhood $\mathcal{O}$ of $\omega(x)$, $0 < \alpha < 1$ and $M > 0$ such that there exists a one-dimensional invariant subbundle $\mathcal{S}$ of $(\overline{O^+(x)} \cap \mathcal{O}) \times V$, with its supplementary subbundle $\mathcal{T}$ (not necessarily invariant) having the following properties*:

(i) $\mathcal{S}|_z \subset P^o \cup (-P^o) \cup \{0\}$ *for each* $z \in \overline{O^+(x)} \cap \mathcal{O}$;

(ii) $\mathcal{T}|_z \cap P = \{0\}$ *for each* $z \in \overline{O^+(x)} \cap \mathcal{O}$;

(iii) *let $\pi_1(z)$ (resp. $\pi_2(z)$) be the projection of V onto $\mathcal{S}|_z$ along $\mathcal{T}|_z$ (resp. onto $\mathcal{T}|_z$ along $\mathcal{S}|_z$). Then*

$$\frac{\|DT^k(z)v\|}{\|\pi_2 DT^k(z)w\|} \leqslant M\alpha^k \frac{\|v\|}{\|w\|} \tag{3.10}$$

for each $z \in \mathcal{O} \cap \overline{O^+(x)}$, $v \in \mathcal{S}|_z$, $0 \neq w \in \mathcal{T}|_z$ and each $k = 1, 2, \cdots$;

(iv) *in particular, $\omega(x) \times V = \mathcal{S}|_{\omega(x)} \bigoplus \mathcal{T}|_{\omega(x)}$ is an invariant Whitney sum decomposition and satisfies the exponential separation, i.e.,*

$$\frac{\|DT^k(z)v\|}{\|DT^k(z)w\|} \leqslant M\alpha^k \frac{\|v\|}{\|w\|} \tag{3.11}$$

for each $z \in \omega(x)$, $v \in \mathcal{S}|_z$, $0 \neq w \in \mathcal{T}|_z$ and each $k = 1, 2, \cdots$.

Proof. Our proof is very similar to that of Proposition 3.2 in Ruelle [26]. In our case, $\overline{O^+(x)}$ is only positively invariant, not invariant, so we cannot obtain our lemma directly from [26, Proposition 3.2]. However, since $\overline{O^+(x)}$ is positively invariant, we can repeat Ruelle's proof to obtain the one-dimensional invariant subbundle $\mathcal{S} = \mathrm{span}(b_z)$, $\|b_z\| = 1$, $z \in \overline{O^+(x)}$ with property (i).

Let V^* be the dual of V and let $(DT^{-1})^*$ be the adjoint bundle map of DT^{-1}. Note that $\omega(x) \subset C^o$ is invariant and T is strongly type-K competitive in C^o, then, by repeating Ruelle's proof, one can find an invariant subbundle $F^* = \mathrm{span}(b_z^*)$ over $\omega(x)$ (here we follow Ruelle's notation). Now choose a small compact neighborhood $\mathcal{O}(\subset C^o)$ of $\omega(x)$ and define $\hat{b}_z^*$ on $\mathcal{O}$ such that $\left\|\hat{b}_z^*\right\| = 1$, the map $z \mapsto \hat{b}_z^*$ is continuous on $\mathcal{O}$ and $\hat{b}_z^* = b_z^*$ for $z \in \omega(x)$. Since $b_z^* \in \mathrm{Int}\, P^*$ for all $z \in \omega(x)$ (see [26, p. 73]), where $P^* = \{\eta \in V^* : \langle \eta, \xi \rangle \geqslant 0 \text{ for all } \xi \in P\}$, one can make $\mathcal{O}$ smaller, if necessary, such that $\hat{b}_z^* \in \mathrm{Int}\, P^*$ for all $z \in \mathcal{O}$. Then there is $\varepsilon > 0$ independent of

$z \in \mathcal{O}$ such that the ball of radius ε centered at $\hat{b}_z^*$ is contained in P^*. Furthermore, we get the one-dimensional subbundle $\hat{F}^*$ over $\mathcal{O}$ (not necessarily invariant) such that $\hat{F}^*|_{\omega(x)} = F^*$, which is invariant. Let $\mathcal{T}$ be the $(n-1)$-dimensional subbundle of $(\overline{O^+(x)} \cap \mathcal{O}) \times V$ orthogonal to $\hat{F}^*$. Then almost same as in Ruelle [26], we define

$$\hat{K}(z) = \{\xi \in V : \langle \hat{b}_z^*, \xi\rangle = 0 \text{ and } b_z + \xi \in P\} \subset \mathcal{T}|_z$$

and

$$T''\xi = \pi_2 DT^{-1}(z)\xi / \left\| DT^{-1}(z) b_z \right\|$$

if $\xi \in V$ and $z \in \mathcal{O} \cap \overline{O^+(x)}$. Then, as in Ruelle's proof (see [26, formula (3.2)]), we find that there is $0 < \alpha < 1$ such that

$$T''\hat{K}(z) \subset \alpha \hat{K}(T^{-1}z)$$

for all $z \in \mathcal{O} \cap \overline{O^+(x)}$. Hereafter, we exactly repeat Ruelle's proof step by step and obtain that

$$\frac{\left\|\pi_2 DT^{-k}(z)w\right\|}{\left\|DT^{-k}(z)v\right\|} \leqslant M\alpha^k \frac{\|w\|}{\|v\|} \tag{3.12}$$

for each $z \in \mathcal{O} \cap \overline{O^+(x)}$ with $z, T^{-1}z, \ldots, T^{-k}z \in \mathcal{O}$, $0 \neq v \in \mathcal{S}|_z$, $0 \neq w \in \mathcal{T}|_z$ and each $k = 1, 2, \cdots$.

Now, we will prove (3.10). Since $\mathcal{O}$ is a neighborhood of $\omega(x)$, it is easy to see that $\mathcal{O} \cap \overline{O^+(x)}$ is positively invariant. Furthermore, we can also prove that $\xi = \pi_2 DT^{-k}(\pi_2 DT^k \xi)$ for every $\xi \in \mathcal{T}$ and $k \in \mathbb{N}$. In fact, for every $\xi \in \mathcal{T}$, one has

$$\xi = DT^{-k}(DT^k\xi) = DT^{-k}(\pi_1 DT^k\xi) + DT^{-k}(\pi_2 DT^k\xi),$$

since $\mathcal{S}$ is invariant, $\pi_2 DT^{-k}(\pi_1 DT^k \xi) = 0$, which implies that $\xi = \pi_2\xi = \pi_2 DT^{-k}$ $(\pi_2 DT^k\xi)$. Hence, given any $z \in \mathcal{O} \cap \overline{O^+(x)}$, $0 \neq v \in \mathcal{S}|_z$, $0 \neq w \in \mathcal{T}|_z$ and $k \in \mathbb{N}$. Let $\tilde{z} = T^k z$, $\tilde{v} = DT^k(z)v$ and $\tilde{w} = \pi_2 DT^k(z)w$. then by (3.12),

$$\frac{\left\|\pi_2 DT^{-k}(\tilde{z})\tilde{w}\right\|}{\left\|DT^{-k}(\tilde{z})\tilde{v}\right\|} \leqslant M\alpha^k \frac{\|\tilde{w}\|}{\|\tilde{v}\|},$$

and hence,

$$\frac{\left\|DT^{k}(z)v\right\|}{\left\|\pi_2 DT^{k}(z)w\right\|} \leqslant M\alpha^k \frac{\|v\|}{\|w\|}.$$

Thus we obtained (3.10). In particular, (3.11) holds on $\omega(x)$. We have completed the proof of Lemma 3.15.

Now we return to the proof of the uniqueness of y_l. It follows from Lemma 3.15(iv) that $\omega(x) \times V = \mathcal{S}|_{\omega(x)} \bigoplus \mathcal{T}|_{\omega(x)}$ is an invariant Whitney sum decomposition and satisfies the condition of exponential separation. By Mierczyński [20, Appendix, Proposition A.3], for each ergodic invariant probability measure m on $\omega(x)$, its Lyapunov exponents $\lambda_1(m), \ldots, \lambda_l(m)$ satisfy $\lambda_1(m) \leq \lambda_j(m) - \alpha$, $j = 2, \ldots, l$.

Inspired by Jiang et al. [46] *we claim that* $\lambda_1(m) \leqslant -\alpha$. Suppose not, then $\lambda_j(m) > 0$ for all $j = 2, \ldots, l$. By the Poincaré Recurrent Theorem [19, Theorem 2.3], the set of recurrent points for $T^{-1}|_{\omega(x)}$ has full m-measure. Since the set of P-regular points also has full m-measure, we can choose z to be a P-regular point in $\omega(x)$ with $z \in \alpha(z)$, which implies that there exists a sequence $n_k \to +\infty$ such that $T^{-n_k} z \to z$. By Pesin's theory, there is a C^1 immersed $(n-1)$-dimensional submanifold D tangent at z to $\mathcal{T}(z)$ and such that $\left\| T^{-n} u - T^{-n} z \right\| \to 0$ as $n \to \infty$ for every $u \in D$. Since the orthogonal projection of $\omega(x)$ along some K-positive vector $e \gg_K 0$ is Lipschitz continuous with a Lipschitz continuous inverse and D is transverse to e, the projection of D on S along e is a Lipschitz homeomorphism onto its image with a Lipschitz inverse. Consequently, there is a k_0 so large that $T^{-n_{k_0}} z = u + \rho e$ for some $u \in D$. Put $p := n_{k_0}$. Assume for definiteness that $\rho \geqslant 0$. We have $T^{-n_k}(T^{-p} z) \to T^{-p} z$ as $k \to \infty$. Since $T^{-p} z \geqslant_K u$ and $\left\| T^{-n_k} u - T^{-n_k} z \right\| \to 0$ as $k \to \infty$, one has $T^{-p} z \geqslant_K z$. But $\omega(x)$ is unordered with respect to $\leqslant_K$, hence $z = T^{-p} z$ and z is a periodic point. Without loss of generality, we assume that p is the prime period of z. Furthermore, we can choose some $\eta > 0$ sufficiently small such that $\lim_{n\to\infty} T^{-pn} z' = z$ for any $z' \in \omega(x)$ with $\left\| z' - z \right\| < \eta$ (In fact, we only need to replace $T^{-p} z$ by z' in the above proof). Therefore, $\omega(x)$ has a nonempty proper attractor $O(z)$ in terms of T^{-p}. By the chain transitivity of $\omega(x)$ (see [12]), $O(z) = \omega(x)$, which is a cycle, and we have a contradiction. Thus, we have proved the claim, i.e., $\lambda_1(m) \leqslant -\alpha$ for all ergodic invariant measures m on $\omega(x)$.

Therefore, it is easy to see that there exists a constant $c > 0$ such that

$$\left\| DT^k(z) b_z \right\| \leqslant c\alpha^k$$

for each $z \in \omega(x)$ and each $k = 1, 2, \cdots$. Using the similar technique in Hirsch and Pugh [13], we can renorm V as $|\cdot|$ such that the new norm $|\cdot|$ is adapted to $\omega(x) \times V$ (cf. [3]), i.e., there exists a constant $0 < \beta < 1$ such that $|DT(z) b_z| < \beta |b_z|$ for all $z \in \omega(x)$. Then, by the continuity of b_z and $DT(z)$ on $\mathcal{O} \cap \overline{O^+(x)}$, one can make $\mathcal{O}$ smaller again, if necessary, such that $|DT(z) b_z| < \beta |b_z|$ for all $z \in \mathcal{O} \cap \overline{O^+(x)}$, which implies that there is a constant c' such that $\left\| DT^k(z) b_z \right\| \leqslant c' \beta^k \left\| b_z \right\| = c' \beta^k$ for each $z \in \mathcal{O} \cap \overline{O^+(x)}$ and each $k = 1, 2, \cdots$. Therefore,

$$\left\| DT^k(z) v \right\| \leqslant c' \beta^k \left\| v \right\| \tag{3.13}$$

for each $z \in \mathcal{O} \cap \overline{O^+(x)}$, $v \in \mathcal{S}|_z$ and each $k = 1, 2, \cdots$. Since (3.10) and (3.13), it follows from [3, Theorems 1–3] that there exists a family of one-dimensional compact strongly stable invariant C^2-manifolds $W^{ss}(z), z \in \mathcal{O} \cap \overline{O^+(x)}$. The manifold $W^{ss}(z)$ is tangent to $\mathcal{S}|_z$ at z.

In the first part of the proof of Proposition 3.14, we have obtained that there exists a $y_0 \in M_i \cap C^o$ such that $\|T^n x - T^n y_0\| \to 0$ as $n \to \infty$ and either $T^n x \ll_K T^n y_0$ or $T^n x \gg_K T^n y_0$ for all $n \in \mathbb{N}$. Assume for definiteness that $T^n x \gg_K T^n y_0$ for all $n \in \mathbb{N}$. Moreover, we set $x_n := T^n x$ and $y_n := T^n y_0$ for simplicity. We claim that the directions of $(x_n - y_n)/\|x_n - y_n\|$ tend, as $n \to +\infty$, to the direction of $\mathcal{S}$. Otherwise, suppose that there exists a subsequence n_k and some $z \in \omega(x)$ such that $x_{n_k} \to z$, $y_{n_k} \to z$ and $(x_{n_k} - y_{n_k})/\left\|x_{n_k} - y_{n_k}\right\| \to u$ as $k \to +\infty$, where $u \notin \mathcal{S}|_z$ with $\|u\| = 1$. Then, for each $j = 1, 2, \ldots,$

$$\frac{x_{j+n_k} - y_{j+n_k}}{\left\|x_{j+n_k} - y_{j+n_k}\right\|} = \frac{\int_0^1 DT(\eta(s, j-1))ds \cdots \int_0^1 DT(\eta(s,0))ds(x_{n_k} - y_{n_k})}{\left\|\int_0^1 DT(\eta(s, j-1))ds \cdots \int_0^1 DT(\eta(s,0))ds(x_{n_k} - y_{n_k})\right\|},$$

where $\eta(s,i) = sx_{i+n_k} + (1-s)y_{i+n_k}$ for $i = 0, \ldots, j-1$. Hence,

$$\frac{x_{j+n_k} - y_{j+n_k}}{\left\|x_{j+n_k} - y_{j+n_k}\right\|} \to \frac{DT^j(z)u}{\left\|DT^j(z)u\right\|} \tag{3.14}$$

as $k \to +\infty$. Since $u \notin \mathcal{S}|_z$, we write $u = u_1 + u_2$, $u_1 \in \mathcal{S}|_z, u_2 \in \mathcal{T}|_z$ and $u_2 \neq 0$. It follows from (3.11) that

$$\frac{DT^j(z)u}{\left\|DT^j(z)u\right\|} = \frac{DT^j(z)u_2\left(1 + \dfrac{DT^j(z)u_1}{DT^j(z)u_2}\right)}{\left\|DT^j(z)u_2\left(1 + \dfrac{DT^j(z)u_1}{DT^j(z)u_2}\right)\right\|} \sim \frac{DT^j(z)u_2}{\left\|DT^j(z)u_2\right\|},$$

as $j \to +\infty$. Here $f(j) \sim g(j)$ means $\lim_{j\to+\infty} f(j)/g(j) = 1$. Then, by Lemma 3.15(ii) and the invariance of $\mathcal{T}$ on $\omega(x)$, one has $\dfrac{DT^j(z)u}{\left\|DT^j(z)u\right\|} \notin P$ for all j sufficiently large. On the other hand, based on our assumption, it is easy to see that $\dfrac{x_{j+n_k} - y_{j+n_k}}{\left\|x_{j+n_k} - y_{j+n_k}\right\|} \in P$ for all $j, n_k \in \mathbb{N}$, a contradiction to (3.14). Thus, we have proved the claim, which implies that $y_n \in M_i \cap W^{ss}(x_n)$ for all n sufficiently large.

Suppose that there exists another $y' \in M_i \cap C^o$ such that $\left\|T^n x - T^n y_0'\right\| \to 0$ as $n \to \infty$ and either $T^n x \ll_K T^n y_0'$ or $T^n x \gg_K T^n y_0'$ for all $n \in \mathbb{N}$. Then, by repeating the proof above, one has $y_n, y_n' \in M_i \cap W^{ss}(x_n)$ for all n sufficiently large. Since $W^{ss}(x_n)$ is tangent to $\mathcal{S}|_{x_n}$ at x_n, it follows from Lemma 3.15(i) that $W^{ss}(x_n)$ is totally ordered with respect to $\ll_K$ in some neighborhood of x_n. Then $y_n, y_n' (= T^n y_0')$ is related by $\ll_K$ for all n sufficiently large. Note that $y_n, y_n' \in M_i$, which contradicts the non-ordering of M_i (see Proposition 3.2(d) and Proposition 3.3(3)). Thus we have proved Proposition 3.14.

Proof of Theorem A. We take $\mathcal{F}$ to be the family as defined before Proposition 3.14. The properties of the elements of $\mathcal{F}$ listed in parts (a)–(d) of Theorem A have been proved in Propositions 3.2 and 3.3.

We now prove (e). Let the trajectory passing through $x \in C^o$ be a persistent one which does not converge to a cycle. It follows from Theorem 3.13 that $\omega(x) \subset M_i$, for some $M_i \in \mathcal{F}$. If $x \in M_i$, then set $y = x$ which fulfills the requirements. If $x \notin M_i$, then, by Proposition 3.14, we can find some non-negative integer l such that for such l there exists a unique $y \in M_i$ such that the forward orbits of y and $T^l x$ are asymptotic, that is, $\|T^{n+l}x - T^n y\| \to 0$ as $n \to +\infty$. We have completed the proof of Theorem A.

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